APPLICATION OF LIE THEORY TO SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS

By

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CERTIFICATE

This is to certify that (Mrs.) Bhupinder Kaur has duly completed her thesis for the degree of Ph.D. of Bundelkhand University, Jhansi and her thesis is upto the mark both in its academic contents and quality of presentation.

I further certify that this work has originally been done by her and she has been working under my supervision since febuary 1981.

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## DECLARATION

I here by state that the work 'Application of Lie Theory to Special Functions of Mathematical Physics.' has been done by me and to the best of my knowledge, a similar work has not been done anywhere so far.

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Bhupinder Kaur

## PREFACE

In the present thesis I have endeavoured to give Lie theoretic approach to various special functions.

The thesis consists of ten chapters each divided into several sections (progressively numbered 1.1, 1.2,...........), Refrences to the literature are given in full at the end of each chapter, In the text they have been reffered to by putting within the square brackets, The result in the text have been numbered serially section and chapterwise e.g. (2.3.6) means sixth result of section three of chapter two.

Bhupinder Kaur

List of Papers, Published, Accepted or Communicated

| S.No. | Title | Chapter | Reffered to Journal | Result |
|---|---|---|---|---|
| 1. | Lie operators and Laguerre Polynomials | II | Ganita, Lucknow | awaited |
| 2. | Some Generating functions for a Polynomial suggested by Laguerre Polynomial -I | III | Under Communication . | |
| 3. | Some Generating Functions for Polynomials suggested by Laguerre Polynomials-II | IV | -do- | |
| 4. | Lie Theory and Hypergeometric functions $_2F_1$ | V | Presented by Dr. P.N. Shrivastava (Guide) at the Annual conference of Indian Mathematical Society held from 27-29 Dec. 1986. | |
| 5. | On certain generating Relations involving classical Polynomials. | VI | Communicated of Jour. of IMS. | awaited |
| 6. | Some Theorems, associated with bilateral generating functions involving Hermite, Laguere and Gegenbauer Polynomials | VII | Journal of Indian Mathematical Society | accepted |
| 7. | Lie operators & Generalised Bessel Polynomials | VIII | Jour. of Maulana Azad College of Technology. | accepted |
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| S.No | Title | Chapter | Reffered to Journal | Result |
|---|---|---|---|---|
| 9. | Dynamical Symmetry Algebra of $F_2$ and Reduction Formulae for Hypergeometric of three variables. | X | Communicated to Journ. of Indian Academic of Mathematics. | awaited |
| 10. | Dynamical Symmetry Algebra of $F_2$ and Generating functions for Different Polynomials. | X | -do- | -do- |

Contents

# CHAPTER - I INTRODUCTION

Special functions of Mathematical Physics which were considered solutions of partial differential equations like wave equation laplace equation, diffusion equation etc. have been studied by different authors in different ways. Harry Bateman (1882-1942) is considered to be one of the foremost mathematician who made a critical study of the subject.

Apart from the application of special functions in Physics and engineering, we find that the study of the subject in theoretical direction is very interesting and has engaged a number of mathematicians for more than a century..

A great ammount of work has been done on the study of classical polynomials namely legendre's polynomial, Hermite polynomial , laguerre polynomial , J acobi polynomial etc. The polynomials and functions obtained after generalising these polynomials have been a fertile field to the research workers in recent days. Another stream in which the study of special functions has been made is the hypergeometric functions. The generalised hypergeometric functions E , G and H functions, and functions of several variable have also been studied and a lot of work is now available on these functions.

With the advancement of knowledge in the field of special functions, it has been the approach of past years research workers to search new and easy approches to establish new results and to give easy methods to obtain certain already known results. In an attempt

in this direction, group theory, has been found to have an important role. This approach is better known as Lie theoretic method. The first significant advancement in this direction was made by L. Weisner (1955 to 1959) [56] [57] [58] who exhibts the group theoretic significance of generating functions for Hypergeometric, Hermite and Bessel functions. Then Willard Miller, Jr. (1968) [40] and E.B. Mcbride (1971) [36] presents weisner's method in a systematic manner and there by lay its firm foundation. Miller' (1968) also extends Weisner's theory further by relating it to factorisation method, originated by Schrodinger and due to its definitive form to Infeld and Hull (1951) [25]. Kalnins, E.G., Manocha H.L, and Miller Jr. W. (1980,1982) [27,28,29] studied Lie Algebraic characterizations of two variable Horn-functions. In the process they evolved a method for obtaining generating functions by expanding a two variable Horn-functions in terms of One-variable hyper-geometric functions. Others, those contributing in this direction are Agarwal B.M., Chatterjea S.K., Pathan M.A. and their team of workers.

In the present thesis the authors have mainly used Weisner's method with the use of Miller's technique, where ever required in obtaining generating functions for a class of functions which are mostly the generalisations of usual classical polynomials. These include - Laguerre polynomials, Konhauser's Bi-Orthogonal polynomials, Gould

Hopper's, second generalisation of Hermite polynomials. For constructing Lie-groups associated with above generalised functions, factorization method has been used. Also Dynamical Symmetry algebra first used by Miller (1973) [38] has been extended to two variable hypergeometric functions and has been used to obtain generating relations and reduction formulas for three variable hypergeometric functions. Recently Srivastava, H.M. and Manocha, H.L. (1984) [42] has also given some account of Lie-algebraic technique for obtaining generating functions is also useful.

The vastness and scattering of the subject makes it difficult to give a comprehensive review of the entire literature, however attempt has been made to deal those aspects which have direct bearing on my work, and done in the present thesis in some detail.

1.1) Generating Functions : The word Generating function was first introduced by Laplace in 1812. The generating function are powerful tools in the investigations of system of polynomials and functions.

We define a generating function for a set of functions $\{f_n(x)\}$ as follows:

Let $G(x,t)$ be a function that can be expanded in powers of t such that

$$G(x,t) = \sum_{n=0}^{\infty} C_n f_n(x) t^n$$

Where $C_n$ is a function of n that may contain the parameters of the set $\{f_n(x)\}$ but independent of x and t. Then $G(x,t)$ is called the generating function of $\{f_n(x)\}$. some of the important classes of generating functions

which are used in the study of special functions are listed below:

(i) $$G(2xt-t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n$$

where $G(x)$ has a formal power series.

(ii) $$e^t\, \psi(xt) = \sum_{n=0}^{\infty} \phi_n(x)\, t^n$$

Where $\psi(u)$ has a formal power series.

(iii) $$A(t)\, \exp\left(\frac{-4xt}{1-t}\right) = \sum_{n=0}^{\infty} y_n(x)\, t^n$$

(iv) $$(1-t)^{-c}\, \psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)\, t^n$$

Where $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n \quad , \quad \gamma_n \neq 0$

(v) $$A(t)\, \psi\{x . H(t)\} = \sum_{n=0}^{\infty} p_n(x)\, t^n .$$

(Baos & Buck) Bos

In which $A(t), \psi(t)$ and $H(t)$ are expressible as power series and many other types.

The types of the above generating functions mentioned are guided by the forms of generating functions obtained for special functions like Hermite Polynomials, Legendre Polynomials, Laguerre Polynomials, Jacobi Polynomials and many other classical polynomials. For a detailed survey of these generating functions One can go through recently published books written by Elna B. Mcbride (1971) [36] and H.M. Srivastava & H.L. Manocha (1984) [42].

Major problem has been search of generating functions for a known set of polynomials and functions. Most of the efforts were made in this direction during the last 30 years only. The contribution of Truesdel, Weisner, Rainville are worth mentioning.

The method of Truesdel is based on the study of F-equation, which is

$$\frac{\partial}{\partial z} F(z,\alpha) = F(z,\alpha+1)$$

The Rainville's method is based on the direct summation techniques, where as Weisner's method is based on the factorization of Ordinary differential equations and their application of group-theory.

In the present days group-theoretic approach has picked up momentum and a good amount of work has been done in this direction. However study of special functions from group-theoretic approach has been detailed in the works of Willard Miller Jr. [38] James D. Talman [52] and N.J. Vilenkin [55] vastly and to some extent Mcbride [36] Srivastava-Manocha [42].

1.2) <u>Group theoretic method</u> : In the present thesis mainly Group-theoretic approach of Weisner has been used. Weisner devised a method for obtaining generating functions for sets of functions, which satisfy certain conditions. Among these functions are Hypergeometric functions ${}_2F_1$ [58], the Hermite Polynomials [57] Bessel functions [56] Laguerre and Gegenbauer polynomials etc. These functions play an important role in Quantum theory. In this method we consider the ordinary differential equation which is satisfied by the set of polynomials or functions under consideration. From this differential equation a partial differential equation is constructed then we construct the non-trivial continuous group transformations (known as Lie group, see Cohn [18] ) under which this partial differential is invariant. For constructing the group of transformations (or operators),

we require a pair of differential recurrence relations, where the subscripts are non-negative integers. There are many methods to obtain these differential recurrence relations still we have a powerful method, known as factorization method, which was originated by Schrodinger and given a definite form by Infeld and Hull [25].

1.3) Factorization Method : Actually Weisner in 1955 was guided by the paper of Infeld-Hull. Infeld-Hull devised a technique of factorization of ordinary differential equation. This technique was developedto solve eigenvalue problems appearing in quantum theory, but has proved to be a very useful tool for studying recurrence formuales obeyed by special functions.

Both in Electromagnetic theory and Quantum theory we are lead to the equations of the type.

1.3.1) $$\frac{d^2y}{dx^2} + r(x,m)\frac{dy}{dx} + \lambda y = 0$$

here $r(x,m)$ is a function which characterises a particular problem. Assuming that m is a non-negative integer, which is gained through separating variables, its value is restricted by the boundary conditions. In most of the cases the boundary conditions require further that $\lambda$ has discrete eigenvalues $\lambda_0, \lambda_1, --- \lambda_\ell, ---$. Thus the typical eigen value problem can be represented by the lattice of points in the $(\ell, m)$ plane. For every point on the lattice there exists a function $y_\ell^m(x)$, some boundary conditions.

The factorization method either treats the original first order differential equation directly or replace the second order differential equation by an equivalent pair

of first order equations of the form:-

1.3.2) $\left\{K(x,m+1)-\frac{d}{dx}\right\}y_{\ell}^{m}=\left[\lambda-L(m+1)\right]^{1/2}y_{\ell}^{m-1}$

1.3.3) $\left\{K(x,m)-\frac{d}{dx}\right\}y_{\ell}^{m}=\left[\lambda-L(m)\right]^{1/2}y_{\ell}^{m-1}$

Infeld-Hull explored only six possibilities and even these six are not independent. For a detailed procedure. One can refer to [25] or for a still modified method to Miller Jr. [40] (Chapter on 'factorization Method').

1.4) Application of Weisner's Method :-

From the above relations (1.3.2) and (1.3.3) We observe, in general, that we get a pair of differential recurrence relations of form

1.4.1) $L^{+}(D,m)\,Y_{\ell}^{m}=\lambda_{m}\cdot Y_{\ell}^{m+1}$

1.4.2) $L^{-}(D,m)\,Y_{\ell}^{m}=\mu_{m}\,Y_{\ell}^{m-1}$

In (1.4.1) & (1.4.2) replace m by $z\frac{\partial}{\partial z}$ and D by $\frac{\partial}{\partial x}$, then, we get a set of partial operators of the form.

1.4.3) $A=z\frac{\partial}{\partial z}$

1.4.4) $J^{+}=y\,L^{+}\left(\frac{\partial}{\partial x},z\frac{\partial}{\partial z}\right)$

1.4.5) $J^{-}=y^{-1}\,L^{-}\left(\frac{\partial}{\partial x},z\frac{\partial}{\partial z}\right)$

Then we examine, on obtaining commutator relations for the above operators, whether $\{1,A,J^{+},J^{-}\}$ form a Lie-group. If these operators form a Lie-group, we derive there extended forms by using Miller's technique or otherwise and proceed, to determine generating relations for the functions $Y_{\ell}^{m}(x)$.Here we further observe that

1.4.6) $A\left\{z^{m}\,Y_{\ell}^{m}(x)\right\}=m.\,z^{m}\,Y_{\ell}^{m}(x)$

1.4.7) $J^{+}\left\{z^{m}\,Y_{\ell}^{m}(x)\right\}=\lambda'_{m}\,z^{m+1}\,Y_{\ell}^{m+1}(x)$

1.4.8) $J^{-}\{z^m Y_l^m(x)\} = \mu'_m z^{m-1} Y_l^{m-1}(x)$

These relations are useful in getting the desired generating functions for $Y_l^m(x)$.

1.5) Dynamical Symmetry Algebra :-

Willard Miller, Jr. [33] constructed 12 raising and lowering operators E which generated a complex 15-dimensional simple Lie Algebra isomorphic to $sl(4)\ (\cong so(6))$ associated with hypergeometric series ${}_2F_1(\alpha, \beta; \gamma; x)$. For this we set

$\Gamma(\gamma-\alpha)$

1.5.1) $f_{\alpha\beta\gamma}(s, u, t, x) = \frac{\Gamma\gamma-\alpha\ \Gamma\alpha}{\Gamma\gamma} \cdot {}_2F_1(\alpha, \beta; \gamma; x) \cdot s^\alpha u^\beta t^\gamma$

Where $\Gamma z$ is gamma function and $\frac{\Gamma\gamma-\alpha\ \Gamma\alpha}{\Gamma\gamma}$ is the normalising factor. Now introduce differential operators $E_{\pm\alpha}$, $E_{\pm\beta}$, $E_{\pm\gamma}$, $E_{\pm\alpha,\pm\gamma}$, $E_{\pm\beta,\pm\gamma}$, $E_{\pm\alpha\pm\beta\pm\gamma}$ defined by

$$E_\alpha = s\,(x\partial x + s\partial s)$$

$$E_{-\alpha} = s^{-1}\{x(1-x)\partial x + t\partial t - s\partial s - xu\partial u\}$$

$$E_\beta = u\,(x\partial x + u\partial u)$$

$$E_{-\beta} = u^{-1}\{x(1-x)\partial x + t\partial t - u\partial u - xs\partial s\}$$

$$E_\gamma = t\,\{(1-x)\partial x + t\partial t - s\partial s - u\partial u\}$$

$$E_{-\gamma} = t^{-1}\{-x\partial x - t\partial t + 1\}$$

$$E_{\alpha\gamma} = st\,\{(1-x)\partial_x - s\partial_s\}$$

$$E_{\beta\gamma} = ut\,\{(1-x)\partial_x - u\partial_u\}$$

$$E_{-\alpha,-\gamma} = s^{-1}t^{-1}\{x(1-x)\partial_x - xu\partial_u + t\partial_t - 1\}$$

$$E_{-\beta,-\gamma} = u^{-1}t^{-1}\{x(1-x)\partial_x - xs\partial_s + t\partial_t - 1\}$$

$$E_{\alpha\beta\gamma} = sut\,\partial x$$

$$E_{-\alpha,-\beta,-\gamma} = s^{-1}u^{-1}t^{-1}\{x(1-x)\partial_x - t\partial_t + xu\partial_u + xs\partial_s - 1\}$$

where $\partial_z = \frac{\partial}{\partial z}$

These operators satisfy the relations

$$E_{\pm\alpha} f_{\alpha\beta\gamma} = \begin{bmatrix} \gamma-\alpha-1 \\ \alpha-1 \end{bmatrix} f_{\alpha\pm 1,\beta,\gamma}$$

$$E_{\pm\beta} f_{\alpha\beta\gamma} = \begin{bmatrix} \beta \\ \alpha-\beta-\gamma \end{bmatrix} f_{\alpha,\beta\pm 1,\gamma}$$

$$E_{\pm\gamma} f_{\alpha\beta\gamma} = \begin{bmatrix} \gamma-\beta \\ \alpha-\gamma+1 \end{bmatrix} f_{\alpha,\beta,\gamma\pm 1}$$

$$E_{\pm\alpha,\pm\gamma} f_{\alpha\beta\gamma} = \begin{bmatrix} \beta-\gamma \\ \alpha-1 \end{bmatrix} f_{\alpha\pm 1,\beta,\gamma\pm 1}$$

$$E_{\pm\beta,\pm\gamma} f_{\alpha\beta\gamma} = \begin{bmatrix} \beta \\ \alpha-\gamma+1 \end{bmatrix} f_{\alpha,\beta\pm 1,\gamma\pm 1}$$

$$E_{\pm\alpha,\pm\beta,\pm\gamma} f_{\alpha\beta\gamma} = \begin{bmatrix} \beta \\ -\alpha+1 \end{bmatrix} f_{\alpha\pm 1,\beta\pm 1,\gamma\pm 1}$$

The upper factor in each bracket is associated with plus sign and lower with minus sign.

Finally, introduce the operators

$$J_\alpha = s\,\partial_s \quad , J_\beta = u\,\partial_u \; , \; J_\gamma = t\,\partial_t$$

Which satisfy the relations

$$J_\alpha f_{\alpha\beta\gamma} = \alpha f_{\alpha\beta\gamma} \quad , \quad J_\beta f_{\alpha\beta\gamma} = \beta f_{\alpha\beta\gamma}$$

$$J_\gamma f_{\alpha\beta\gamma} = \gamma f_{\alpha\beta\gamma}$$

i.e., $f_{\alpha\beta\gamma}$ is a simultaneous eigenfunctions of $J_\alpha, J_\beta, J_\gamma$. Note that these operators satisfy the commutator relations:

$$[E_\gamma, E_\alpha] = E_{\alpha\gamma} \quad , \quad [E_\alpha, E_\beta] = 0$$

$$[E_\alpha, E_{-\alpha}] = 2J_\alpha - J_\gamma$$

$$[E_\gamma, E_{-\gamma}] = 2J_\gamma - J_\alpha - J_\beta - 2I$$

Where $[A,B] = AB - BA$ and $I$ is the identity operator. Here $J_\alpha, J_\beta$ and $J_\gamma$ do not belong to $sl(4)$ but they belong to the 16-dimensional Lie algebra $gl(4)$

$\simeq sl(4) \oplus (I)$. Thus the 12 operators E, together with the four operators $J_\alpha, J_\beta, J_\gamma, I$ form a basis for $gl(4)$

Since $sl(4)$ is the Lie algebra generated by all raising and lowering operators, Miller has called it Dynamical Symmetry Algebra of ${}_2F_1$ in anology with quantum theory. Use of these operators has been made by Miller to derive certain generating relations for ${}_2F_1$ .

Being motivated by the above BM Agarwal and Renu Jain [1] used the Dynamical Symmetry algebra of ${}_2F_1$ to derive certain results associated with Jacobi polynomials. Also Renu Jain [2] has derived few generating functions & reduction formulas for generalised hypergeometric functions.

Brief Survey :-

Chapter II is devoted to the use of Lie theory to obtain some generating functions for Laguerre polynomials $L_n^\alpha(x)$ when both $\alpha$ and n vary.

Chapter III is devoted to construct Lie operators associated with konhauser's biorthogonal polynomial $Y_n^\alpha(x;K)$ and to derive some generating functions for it.

K

Chapter IV is devoted to construct Lie operators associated with Konhauser's second biorthogonal polynomial $Z_n^\alpha(x;K)$ and to derive some generating functions for it.

Chapter V is devoted to apply Lie theory to obtain generating functions for ${}_2F_1(\alpha, \beta; c+n; x)$ by varying denominator parameter c.

Chapter VI and VII are devoted to use Lie operator to obtain extension of bilinear and bilateral generating functions for classical polynomials and their generalisation.

Chapter VIII is devoted to obtain Lie operators associated with the generalised Bessel function $y_n(x,a,b)$ of Krall & Frink and to derive some generating functions for $y_n(x,a,b)$.

Chapter IX is devoted to construct Lie operators associated with second generalisation of Hermite polynomials, $g_n^r(x,h)$ of Gould-Hopper and to use these operators to obtain some generating relations for $g_n^r(x,h)$.

Chapter X is devoted to the construction of Dynamical Symmetry algebra of hypergeometric function of two variables $F_2$ and to derive some generating relations and reduction formulae for hypergeometric functions of three variables.

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# CHAPTER II

## Lie Operators and Laguerre Poly nomials

2.1) **Introduction** : Many authors [1] [3] have used group-theoretic methods to obtain certain generating relations for the Laguerre polynomials $L_n^{\alpha}(x)$ defined by Rodrigue's type formula.

$$2.1.1) \quad L_n^{(\alpha)}(x) = \frac{x^{-n-\alpha-1}}{\lfloor n} (x^2 D)^n \left[x^{\alpha+1} . e^{-x}\right]$$

Where $\alpha$ is an arbitrary parameter & n is a positive integer. On keen observation, It's found that the Lie operators used by above authors either bring change in $\alpha$ keeping n unchanged, or change in n, keeping $\alpha$ unchanged. Here in the present chapter the authors have described thoses operators which bring change in n and $\alpha$ together and have obtained below some generating functions for $L_n^{\alpha}(x)$ by group theoretic method.

2.2) Lie operators associated by $L_n^{\alpha}(x)$ :-

The differential equation satisfied by $L_n^{\alpha}(x)$ is given as

$$2.2.1) \quad \left[x D^2 + (\alpha - x + 1) D + n\right] L_n^{\alpha}(x) = 0 ,$$

$$D \equiv \frac{d}{dx}$$

From (2.2.1) we obtain following differential recurrence relations [4]

$$2.2.2) \quad D\, L_n^{\alpha}(x) = - L_{n-1}^{\alpha+1}(x)$$

$$2.2.3) \quad (xD + \alpha - x) L_n^{\alpha}(x) = (n+1)\, L_{n+1}^{\alpha-1}(x)$$

Replacing n by $y \frac{\partial}{\partial y}$, D by $\frac{\partial}{\partial x}$, and $\alpha$ by $t \frac{\partial}{\partial t}$ in (1.2.1), consider the partial differential equation.

$$2.2.4) \quad x \frac{\partial^2 u}{\partial x^2} + t \frac{\partial^2 u}{\partial t \partial x} + (1-x) \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

This equation is satisfied by

$$u(x,y,t) = t^{\alpha} y^{n} L_n^{\alpha}(x) \quad 1$$

for brevity we express as

$$L = x\frac{\partial^2}{\partial x^2} + t\frac{\partial^2}{\partial t\,\partial x} + (1-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

So that (1.2.4) can be expressed as

2.2.4a) $$L u(x,y,t) = 0$$

Now consider the differential operators

2.2.5) $$A = y\frac{\partial}{\partial y} \qquad B = -y^{-1}t\frac{\partial}{\partial x}$$

$$\mathcal{C} = t^{-1}yx\frac{\partial}{\partial x} + y\frac{\partial}{\partial t} - xyt^{-1}$$

$$D = t\frac{\partial}{\partial t}$$

Then

2.2.6) $$L = A - B\mathcal{C} + 1 = A + \alpha^{-1}D - B\mathcal{C}$$

Also we have

2.2.7) $$A\left\{t^{\alpha}y^{n}L_n^{\alpha}(x)\right\} = n t^{\alpha}y^{n}L_n^{\alpha}(x)$$

2.2.8) $$B\left\{t^{\alpha}y^{n}L_n^{\alpha}(x)\right\} = t^{\alpha+1}y^{n-1}L_{n-1}^{\alpha+1}(x)$$

2.2.9) $$\mathcal{C}\left\{t^{\alpha}y^{n}L_n^{\alpha}(x)\right\} = (n+1)t^{\alpha-1}y^{n+1}L_{n+1}^{\alpha-1}(x)$$

2.2.10) $$D\left\{t^{\alpha}y^{n}L_n^{\alpha}(x)\right\} = \alpha t^{\alpha}y^{n}L_n^{\alpha}(x)$$

The commutator relations satisfied by $A.B,\mathcal{C}$ and D are

2.2.11) $$[A,B] = -B \quad , \quad [A,\mathcal{C}] = \mathcal{C}$$

$$[B,\mathcal{C}] = 1 \quad , \quad [D,B] = B$$

$$[D,\mathcal{C}] = -\mathcal{C}$$

Clearly, we have observe that $\{1,A,B,\mathcal{C}\}$ and $\{1,D,B,\mathcal{C}\}$ ~~generate Lie groups.~~ generate Lie groups.

To find extended forms of the transformation groups generated by A we use the group theoretic method by Miller (2). For that we have to solve following equations.

$$\frac{\partial y(a)}{\partial a} = y(a)$$

$$\int \frac{\partial y(a)}{y(a)} = \int \partial a + K$$

$$\log y(a) = a + K$$

When $a=0$, $y(0)=y$, $k=\log y$

Hence
$$y(a) = ye^{a}$$

2.2.12) $(\exp aA)\, f(x,y,t) = f(x, ye^{a}, t)$

To find extended forms of the transformation group generated by B, we have to solve following equations.

$$\frac{\partial x(b)}{\partial b} = -\frac{t}{y}$$

$$x(b) = -\frac{bt}{y} + K$$

When $b=0$ $x(0)=x$, $K=x$

Hence
$$x(b) = x - \frac{bt}{y}$$

2.2.13) $(\exp bB)\, f(x,y,t) = f(x - bty^{-1}, y, t)$

To find extended forms of the transformation group generated by C we have to solve following equations

(i) $$\frac{\partial t(c)}{\partial c} = y$$

$$\int \partial t(c) = y\int \partial c + K$$

$$t(c) = yc + K.$$

When $c=0$ $t(0)=t$, $K=t$

$$t(c) = t+yc$$

and

(ii) $$\frac{\partial x(c)}{\partial c} = \frac{y\, x(c)}{t(c)}$$

$$\frac{\partial x(c)}{x(c)} = \frac{y}{t+yc}\,\partial c$$

$$\log x(c) = \log(t+yc) + K$$

when $c = 0$, $\quad x(0)=x$, $\quad K=\log\frac{x}{t}$

$$x(c) = \frac{x}{t}(t+yc) = x + \frac{xyc}{t}$$

(iii)
$$\frac{\partial v(c)}{\partial c} = -\frac{x(c).\,y}{t(c)}\,.\,v(c)$$

$$\int\frac{\partial v(c)}{v(c)} = -\int\frac{y(x+xyct^{-1})}{t+yc}\,.\,\partial c + K$$

$$\log v(c) = -\frac{yx}{t}c + K$$

When $c=0$, $\quad v(0)=1$, $\quad K = 0$

Hence

$$v(c) = e^{-\frac{cyx}{t}}$$

2.2.14) $$(\exp cG)\,f(x,y,t) = e^{-\frac{cyx}{t}}\,.$$
$$\cdot f(x+xyct^{-1},\,y,\,t+yc)$$

Now to find extended forms of the transformation groups generated by D we have to solve following equations.

$$\frac{\partial t(d)}{\partial d} = t(d)$$

$$\int\frac{\partial t(d)}{t(d)} = \int\partial d + K$$

$$\log t(d) = d + K$$

when $d=0$, $\quad t(0)=t$, $\quad K=\log t$

Hence

$$t(d) = te^{d}$$

2.2.15) $$(\exp d\,D)\,f(x,y,t) = f(x,y,te^{d})$$

2.3) <u>Generating functions of functions annulled by conjugates of (A-n) and (D-α)</u> :-

From (1.2.13) and (1.2.14) we have

2.3.1) $$(\exp bB)(\exp c\mathcal{C})\, f(x,y,t) = \exp\left[-cy(x-bty^{-1})t^{-1}\right] \cdot f\left\{(x-bty^{-1})(1+yct^{-1}),\, y,\, t+yc\right\}$$

Put $S = e^{bB+c\mathcal{C}}$ then $SAS^{-1}$ is conjugate of A and $SDS^{-1}$ is conjugate of D and $G(x,y,t)$ is annulled by L, $S(A-n)S^{-1}$ and $S(D-\alpha)S^{-1}$, where

2.3.2) $$G(x,y,t) = e^{bB+c\mathcal{C}}\left\{t^{\alpha} y^{n} L_n^{\alpha}(x)\right\} = \exp\left[-cyt^{-1}(x-bty^{-1})\right](t+yc)^{\alpha} y^{n} \cdot L_n^{\alpha}\left\{(x-bty^{-1})(1+yct^{-1})\right\}$$

Now consider the following cases

Case I Put c=0, b=1 in (2.3.2) then it reduces to

2.3.3) $$e^{B}\left(t^{\alpha} y^{n} L_n^{\alpha}(x)\right) = t^{\alpha} y^{n} L_n^{\alpha}\left(x-\frac{t}{y}\right)$$

Also

$$e^{B}\left(t^{\alpha} y^{n} L_n^{\alpha}(x)\right) = \sum_{m=0}^{\infty} \frac{(B)^m}{\lfloor m} \left\{t^{\alpha} y^{n} L_n^{\alpha}(x)\right\}$$

$$= \sum_{m=0}^{\infty} \frac{(B)^{m-1}}{\lfloor m} \left\{t^{\alpha+1} y^{n-1} L_{n-1}^{\alpha+1}(x)\right\}$$

$$= \sum_{m=0}^{\infty} \frac{(B)^{m-m}}{\lfloor m} \left\{t^{\alpha+m} y^{n-m} L_{n-m}^{\alpha+m}(x)\right\}$$

2.3.4) Thus $e^{B}\left\{t^{\alpha} y^{n} L_n^{\alpha}(x)\right\} = \sum_{m=0}^{\infty} \frac{1}{\lfloor m} \left\{t^{\alpha+m} y^{n-m} L_{n-m}^{\alpha+m}(x)\right\}$

Thus equating (2.3.3) and (2.3.4) and replacing $t/y$ by z, we get generating relations as

2.3.5) $$\sum_{m=0}^{\infty} \frac{z^m}{\lfloor m} L_{n-m}^{\alpha+m}(x) = L_n^{\alpha}(x-z)$$

Which is Taylor's expansion.

Case II-Put c=1, b=0, in (2.3.2) then it reduces to

2.3.6) $$e^{\mathcal{G}}\{t^{\alpha}y^{n}L_n^{\alpha}(x)\} = \exp\left(-\frac{xy}{t}\right)\cdot (t+y)^{\alpha}y^{n}L_n^{\alpha}\{x(1+yt^{-1})\}$$

Also

$$e^{\mathcal{G}}\{t^{\alpha}y^{n}L_n^{\alpha}(x)\} = \sum_{m=0}^{\infty}\frac{(\mathcal{G})^{m}}{\lfloor m}t^{\alpha}y^{n}L_n^{\alpha}(x)$$

$$= \sum_{m=0}^{\infty}\frac{(\mathcal{G})^{m-1}}{\lfloor m}(n+1)t^{\alpha-1}y^{n+1}L_{n+1}^{\alpha-1}(x)$$

$$= \sum_{m=0}^{\infty}\frac{(\mathcal{G})^{m-m}}{\lfloor m}\{(n+1)\cdots(n+m)\}\cdot t^{\alpha-m}y^{n+m}\cdot L_{n+m}^{\alpha-m}(x)$$

2.3.7) $$= \sum_{m=0}^{\infty}\binom{n+m}{m}t^{\alpha-m}y^{n+m}L_{n+m}^{\alpha-m}(x)$$

Equating (2.3.6) and (2.3.7) and replacing $y/t$ by z, we get

2.3.8) $$\sum_{m=0}^{\infty}\binom{m+n}{m}z^{m}L_{n+m}^{\alpha-m}(x) = (1+z)^{\alpha}\exp(-xz)\cdot L_n^{\alpha}\{x(1+z)\}$$

Case III Put C=1 in (2.3.2), then we get b≠0

2.3.9) $$e^{b\mathcal{B}+\mathcal{G}}\{t^{\alpha}y^{n}L_n^{\alpha}(x)\}$$

$$= \exp[-yt^{-1}(x-bty^{-1})](t+y)^{\alpha}y^{n}\cdot L_n^{\alpha}\{(x-bty^{-1})(1+yt^{-1})\}$$

Also

$$e^{b\mathcal{B}+\mathcal{G}}\{t^{\alpha}y^{n}L_n^{\alpha}(x)\}$$

$$= e^{b\mathcal{B}}\left[\sum_{m=0}^{\infty}\binom{m+n}{m}t^{\alpha-m}y^{n+m}L_{n+m}^{\alpha-m}(x)\right]$$

$$= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{b^{s}}{\lfloor s}\binom{m+n}{m}t^{\alpha-m+s}y^{n+m-s}L_{n+m-s}^{\alpha-m+s}(x)$$

2.3.10)

Thus equating (2.3.9) and (2.3.10) and after adjustments of parameters, we get the generating relation as.

$$\sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{m} \frac{b^s}{s!} z^{m-s} L_{n+m-s}^{\alpha-m+s}(x) = \exp\left[-z(x-bz^{-1})\right](1+z)^{\alpha} \cdot L_n^{\alpha}\left\{(x-bz^{-1})(1+z)\right\} \tag{2.3.11}$$

The generating functions which are annulled by L and operators not conjugate to A require consideration, However, these are not discussed here.

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# CHAPTER III

## Some Generating Functions for a Polynomials Suggested by Laguerre Polynomial -I.

3.1) INTRODUCTION : Konhauser [1] has considered two classes of polynomials $Y_n^\alpha(x;K)$ and $Z_n^\alpha(x;K)$ where $Y_n^\alpha(x;K)$ is a polynomial in $x$ while $Z_n^\alpha(x;K)$ is a polynomial in $x^K$, $\alpha > -1$ and $K=1.2.3........$

An explicit expression for the polynomials $Y_n^\alpha(x;K)$ was given by Carlitz [2] as

3.1.1) $$Y_n^\alpha(x;K) = \frac{1}{\lfloor n}\sum_{i=0}^{n}\frac{x^i}{\lfloor i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{K}\right)_n$$

Where $(\lambda)_n$ is Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma\lambda+n}{\Gamma\lambda}$$

A Rodrigues formula for $Y_n^\alpha(x;K)$ is given by [3].

3.1.2) $$Y_n^\alpha(x;K) = \frac{x^{-Kn-\alpha-1}}{K^n\lfloor n}(x^{K+1}D)^n\left[x^{\alpha+1}\cdot e^{-x}\right]$$

With the help of differential recurrence relations given in [3], we obtain differential recurrence relations for $Y_n^\alpha(x;K)$ as

3.1.3) $$\left\{(1-D)^K-1\right\}Y_n^\alpha(x;K) = Y_{n-1}^{\alpha+K}(x;K)$$

3.1.4) $$(xD+\alpha+1-x-K)\,Y_n^\alpha(x;K) = (n+1)K\cdot Y_{n+1}^{\alpha-K}(x;K)$$

from (3.1.3) and (3.1.4) we get the following differential equations for $Y_n^\alpha(x;K)$

3.1.5) $$\left[\left\{(1-D)^K-1\right\}(xD+\alpha+1-x-K)-K(n+1)\right]\cdot Y_n^\alpha(x;K) = 0$$

In the present chapter, we use the group theoretic method to obtain certain generating functions for

$Y_n^\alpha(x;K)$ ① 

3.2) **Lie Operators associated with $Y_n^\alpha(x;K)$ :-**

Replacing $\alpha$ by $t\frac{\partial}{\partial t}$ and $D$ by $\frac{\partial}{\partial x}$ and $n$ by $y\frac{\partial}{\partial y}$ in (3.1.5) we get the partial differential equation satisfied by $u(x,y) = t^\alpha y^n Y_n^\alpha(x;K)$ as

3.2.1) $$Lu(x,y) = \left[\{(1-D)^K - 1\}(xD + t\tfrac{\partial}{\partial t} + 1 - x - K) - K - Ky\tfrac{\partial}{\partial y}\right] \cdot u(x,y) = 0$$

Now consider the following differential operators

$$A = y\frac{\partial}{\partial y}$$

3.2.2) $$B = \frac{t^K}{y}\left(1 - \frac{\partial}{\partial x}\right)^K - \frac{t^K}{y}$$

$$C = t^{-K} y \left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + 1 - x - K\right)$$

Then

3.2.3) $$(-1)^K L \equiv BC - K(A+1)$$

3.2.4) $$A\{y^n t^\alpha Y_n^\alpha(x;K)\} = n y^n t^\alpha Y_n^\alpha(x;K)$$

$$B\{y^n t^\alpha Y_n^\alpha(x;K)\} = y^{n-1} t^{\alpha+K} Y_{n-1}^{\alpha+K}(x;K)$$

$$C\{y^n t^\alpha Y_n^\alpha(x;K)\} = (n+1)K t^{\alpha-K} y^{n+1} \cdot Y_{n+1}^{\alpha-K}(x;K)$$

The commutator relations are

3.2.5) $$[A,B] = -B, \quad [A,C] = C$$

$$[B,C] = K$$

These commutator relations show that $1, A, B, C$ generate a Lie group transformation we express the extended forms of the group generated by $A$ by solving the following equations.

$$\frac{\partial y(a)}{\partial a} = y(a)$$

$$\int \frac{\partial y(a)}{y(a)} = \int \partial a + K$$

$$\log y(a) = a + K$$

When $a=0$. $y(0)=y$ $K=\log y$

$$y(a) = ye^a$$

3.2.6) Thus $$\exp aA\{f(x,y,t)\} = f(x, ye^a, t)$$

The standard Lie-Theoretic technique is not applicable to find the extended forms of the group generated

by B. for this we consider

$$e^{bB}\{y^n t^\alpha f(x)\} = e^{t^K \frac{b}{y}\{(1-D)^K - 1\}}\{y^n t^\alpha f(x)\} \quad \text{where } D = \frac{\partial}{\partial x}$$

$$= y^n t^\alpha e^{-t^K \frac{b}{y}} \cdot e^{t^K \frac{b}{y}(1-D)^K} f(x)$$

$$= y^n t^\alpha e^{-t^K \frac{b}{y}} \cdot e^{t^K \frac{b}{y}(1-D)^K}\left[e^x \cdot e^{-x} f(x)\right]$$

$$= y^n t^\alpha e^{-t^K \frac{b}{y}} \cdot e^x e^{t^K \frac{b}{y}(1-D-1)^K}\left[e^{-x} f(x)\right]$$

$$= y^n t^\alpha e^{-t^K \frac{b}{y}} e^x e^{t^K \frac{b}{y}(-D)^K}\left[e^{-x} f(x)\right]$$

We have used the operational relations

$$\phi(D)\left[e^x f(x)\right] = e^x\left[\phi(D+1) f(x)\right]$$

We proceed as

$$= e^x t^\alpha y^n e^{-t^K b/y}\left[e^{t^K \frac{b}{y} D_z^K}\{e^z f(-z)\}\right]$$

$$\text{put } x = -z, \quad D_z = \frac{\partial}{\partial z}$$

$$= e^x t^\alpha y^n e^{-t^K b/y}\left[f(-D_u) e^{\frac{b}{y} u^K t^K} e^{uz}\right]_{u=1}$$

$$= e^x t^\alpha y^n e^{-t^K b/y}\left[f(-D_u) e^{uz + \frac{b}{y}(ut)^K}\right]_{u=1}$$

Thus

3.2.7) $$\exp bB\{f(x,y,t)\} = e^{x - t^K \frac{b}{y}}\left[f(-D_u, y, t) e^{\frac{b}{y}(ut)^K - ux}\right]_u$$

$$\text{where } D_u = \frac{\partial}{\partial u}$$

To obtain (3.2.7) we make use of the operational relation, by Gould and Hopper [7].

3.2.8) $$e^{hD_x^r}\{f(x) \cdot e^{tx}\} = f(D_t)\{e^{ht^r} \cdot e^{tx}\}$$

To obtain extended form of the group generated by $\mathcal{C}$ we use usual Miller's [4] technique and find $e^{c\mathcal{C}} f(x,y,t)$ by solving following equations.

(i) $$\frac{\partial t(c)}{\partial c} = \{t(c)\}^{-K+1} \cdot y.$$

$$\int \frac{\partial t(c)}{\{t(c)\}^{-K+1}} = \int y \cdot \partial c + R$$

$$\frac{\{t(c)\}^K}{K} = yc + R$$

When $c=0$, $t(0)=t$, $R=\frac{t^K}{K}$

$$t(c) = \{Kyc + t^K\}^{1/K}$$

$$t(c) = t\{1+Kyct^{-K}\}^{1/K}$$

(ii) $$\frac{\partial x(c)}{\partial c} = yt^{-K}(1+Kyct^{-K})^{-1} . x(c)$$

$$\int \frac{\partial x(c)}{x(c)} = yt^{-K}\int \frac{\partial c}{1+Kyct^{-K}} + R$$

$$\log x(c) = \frac{1}{K}\log(1+Kyct^{-K}) + R$$

When $c=0$, $x(0)=x$, $R=\log x$

$$x(c) = x(1+Kyct^{-K})^{1/K}$$

(iii) $$\frac{\partial v(c)}{\partial c} = v(c)\left[\{1-x(c)-K\}y\{t(c)\}^{-K}\right]$$

$$\frac{\partial v(c)}{v(c)} = \{1-x(1+Kyct^{-K})^{1/K}-K\}. yt^{-K} . (1+Kyct^{-K})^{-1} . \partial c$$

$$\int \frac{\partial v(c)}{v(c)} = \int \frac{yt^{-K}}{1+Kyct^{-K}} . \partial c - \int x y t^{-K}(1+Kyct^{-K})^{\frac{1}{K}-1} . \partial c - \int Kyt^{-K} . \frac{\partial c}{1+Kyct^{-K}}$$

$$\log v(c) = \frac{1}{K}\log(1+Kyct^{-K}) - x(1+Kyct^{-K})^{1/K} - \log(1+Kyct^{-K}) + R$$

When $c=0$, $v(0)=1$ $R=x$

$$v(c) = (1+Kyct^{-K})^{1/K-1} . e^{x - x(1+Kyct^{-K})^{1/K}}$$

Hence

3.2.9) $$e^{cG} f(x,y,t) = (1+Kyct^{-K})^{1/K-1}\exp\left[x - x(1+Kyct^{-K})^{1/K}\right] . f\{x(1+Kyct^{-K})^{1/K}, y, t(1+Kyct^{-K})^{1/K}\}$$

3.3) <u>Generating Functions annulled by Conjugates of (A-n):-</u>

We see that $u(x,y) = y^n t^\alpha Y_n^\alpha(x;K)$ are solutions of the simultaneous equations $Lu=0$. and $Au=nu$ for arbitrary n. Now

3.3.1) $$e^{bB+c\mathcal{C}}\{y^n t^\alpha Y_n^\alpha(x;K)\}$$
$$= y^n t^\alpha (1+Kcyt^{-K})^{\frac{\alpha-K+1}{K}} \cdot e^{x-\frac{b}{y}t^K(1+Kcyt^{-K})}$$
$$\cdot\left[Y_n^\alpha(-Du;K)\, e^{\frac{b}{y}(ut)^K}(1+Kcyt^{-K})\right.$$
$$\left.\cdot\, e^{-ux(1+Kcyt^{-K})^{1/K}}\right]_{u=1}$$
$$= G(x,y)$$

Put $S = e^{bB+c\mathcal{C}}$ then $SAS^{-1}$ is a conjugate of $A$ and $G(x,y)$ is annulled by L and $S(A-n)S^{-1}$

Now we consider the following cases

Case I $\quad b=0, \quad c\neq 0,$

Then (3.3.1) reduces to

3.3.2) $$e^{c\mathcal{C}}[y^n t^\alpha Y_n^\alpha(x;K)]$$
$$= y^n t^\alpha (1+Kyct^{-K})^{\frac{\alpha-K+1}{K}} \cdot$$
$$\cdot\, e^{\{x - x(1+Kyct^{-K})^{1/K}\}}$$
$$\cdot\, Y_n^\alpha\{x(1+Kcyt^{-K})^{1/K};K\}$$

Also

3.3.3) $$e^{c\mathcal{C}}[y^n t^\alpha Y_n^\alpha(x;K)] = \sum_{m=0}^{\infty}\frac{c^m(\mathcal{C})^m}{\lfloor m}[y^n t^\alpha Y_n^\alpha(x;K)]$$
$$= \sum_{m=0}^{\infty}\frac{c^m(\mathcal{C})^{m-1}}{\lfloor m}(n+1)K\, t^{\alpha-K} y^{n+1} Y_{n+1}^{\alpha-K}(x;K)$$
$$= \sum_{m=0}^{\infty}\frac{c^m}{\lfloor m}(\mathcal{C})^{m-m}\{(n+1)(n+2)\cdots(n+m)\}\cdot$$
$$\cdot K^m t^{\alpha-mK} y^{n+m} Y_{n+m}^{\alpha-mK}(x;K)$$
$$= \sum_{m=0}^{\infty} c^m\binom{n+m}{m}K^m t^{\alpha-mK} y^{n+m} Y_{n+m}^{\alpha-mK}(x;K)$$
$$= \sum_{m=0}^{\infty}(cKyt^{-K})^m\binom{n+m}{m} y^n t^\alpha Y_{n+m}^{\alpha-mK}(x;K)$$

Equating the two values and after appropriate adjustments we get the generating relation as

3.3.4) $$(1+Kcyt^{-K})^{\frac{\alpha-K+1}{K}} e^{\{x - x(1+Kcyt^{-K})^{1/K}\}} \cdot y_n^{\alpha}\{x(1+Kcyt^{-K})^{1/K} ; K\}$$
$$= \sum_{m=0}^{\infty} (Kcyt^{-K})^m \binom{n+m}{m} y_{n+m}^{\alpha-mK}(x;K)$$

replace $(1+Kcyt^{-K})^{1/K} \longrightarrow z$ then we get

3.3.4(a) $$z^{\alpha-K+1} e^{x(1-z)} y_n^{\alpha}(xz;K) = \sum_{m=0}^{\infty} \binom{n+m}{m} (z^K-1)^m y_{n+m}^{\alpha-mK}(x;K)$$

for K=1, it gives

3.3.4(b) $$z^{\alpha} e^{x(1-z)} L_n^{\alpha}(xz) = \sum_{m=0}^{\infty} \binom{m+n}{m} (z-1)^m L_{n+m}^{\alpha-m}(x)$$

<u>Case II</u>:- $c=0$, $b\neq 0$

Then (3.3.1) changes to

3.3.5) $$e^{bB}\left[y^n t^{\alpha} y_n^{\alpha}(x;K)\right] = e^{x - \frac{b}{y}t^K} y^n t^{\alpha} \cdot \left[y_n^{\alpha}(-D_u;K) \cdot e^{\frac{b}{y}(ut)^K - ux}\right]_{u=1}$$

Also

3.3.6) $$e^{bB}\left[y^n t^{\alpha} y_n^{\alpha}(x;K)\right] = \sum_{m=0}^{\infty} \frac{b^m}{\lfloor m} B^m \left[y^n t^{\alpha} y_n^{\alpha}(x;K)\right]$$
$$= \sum_{m=0}^{\infty} \frac{b^m}{\lfloor m} B^{m-1}\left[y^{n-1} t^{\alpha+K} y_{n-1}^{\alpha+K}(x;K)\right]$$
$$= \sum_{m=0}^{\infty} \frac{b^m}{\lfloor m} B^{m-m} y^{n-m} t^{\alpha+mK} y_{n-m}^{\alpha+mK}(x;K)$$

Equating the two values and after appropriate adjustments we get a generating relation as

3.3.7) $$e^{x-\frac{b}{y}t^K}\left[y_n^{\alpha}(-Du;K)\,.\,e^{\frac{b}{y}(ut)^K-ux}\right]_{u=1}$$

$$=\sum_{m=0}^{\infty}\left(\frac{b}{y}t^K\right)^m\frac{1}{\underline{|m}}\,y_{n-m}^{\alpha+mK}(x;K)$$

for K=1, (3,3.7) converts into a generating relation for Laguerre Polynomials

3.3.8) $$e^{x-\frac{b}{y}t}\left[e^{u(\frac{b}{y}t-x)}\,.\,L_n^{\alpha}\left\{-\left(\frac{b}{y}t-x\right)\right\}\right]$$

$$=\sum_{m=0}^{\infty}\left(\frac{b}{y}t\right)^m\frac{1}{\underline{|m}}\,L_{n-m}^{\alpha+m}(x)$$

Using the relations given below

$$\left[f(D_t)\,.\,e^{t\phi(x,y)}\right]_{t=1}=\left[e^{t.\phi(x,y)}\,.\,f\{\phi(x,y)\}\right]_{t=1}$$

<u>Case III</u>:- b $\neq 0$, c$\neq 0$

Let C=1, then (3.3.1) changes to

3.3.9) $$e^{bB}\,.\,e^{\mathcal{C}}\,.\left\{y^n t^{\alpha}\,y_n^{\alpha}(x;K)\right\}$$

$$=y^n t^{\alpha}(1+Kyt^{-K})^{\frac{\alpha-K+1}{K}}\,.$$

$$.\,e^{x-\frac{b}{y}t^K(1+Kyt^{-K})}$$

$$.\left[y_n^{\alpha}(-Du;K)\,.\,e^{\frac{b}{y}(ut)^K(1+Kyt^{-K})-ux(1+Kyt^{-K})^{1/K}}\right]_{u=1}$$

3.3.10) But $$e^{bB}\,.\,e^{\mathcal{C}}\left\{y^n t^{\alpha}\,y_n^{\alpha}(x;K)\right\}$$

$$=\sum_{m=0}^{\infty}\sum_{S=0}^{\infty}K^m\binom{n+m}{m}\frac{b^S}{\underline{|S}}\,t^{\alpha-Km+KS}\,y^{m+n-S}\,.$$

$$.\,y_{m+n-S}^{\alpha-mK+KS}(x;K)$$

In the same way as we have done in (3.3.3) and (3.3.6).

Equating the two values and after appropriate

adjustments we get a generating relation as

3.3.11) $$(1+Kyt^{-K})^{\frac{\alpha-K+1}{K}} \cdot e^{x-\frac{b}{y}t^K(1+Kyt^{-K})} \cdot \left[Y_n^\alpha(-Du;K) \cdot e^{\frac{b}{y}(ut)^K(1+Kyt^{-K})-ux(1+Kyt^{-K})^{1/K}}\right]_{u=1}$$
$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} K^m \frac{b^s}{\lfloor s} \binom{n+m}{m} t^{Ks-Km} y^{m-s}\, Y_{m+n-s}^{\alpha-mK+Ks}(x;K)$$

for $K=1$, (3.3.11) gives

3.3.12) $$(1+yt^{-1})^\alpha\, e^{x-\frac{b}{y}t(1+yt^{-1})} \cdot \left[L_n^\alpha(-Du) \cdot e^{\frac{b}{y}ut(1+yt^{-1})-ux(1+yt^{-1})}\right]_{u=1}$$
$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} \frac{b^s}{\lfloor s} \binom{n+m}{m} t^{s-m} y^{m-s}\, L_{m+n-s}^{\alpha-m+s}(x)$$

On simplifying more it turns to

3.3.13) $$(1+yt^{-1})^\alpha\, e^{-xyt^{-1}}\, L_n^\alpha\left(-\frac{b}{y}t-b+x+xyt^{-1}\right)$$
$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} \frac{b^s}{\lfloor s} \binom{n+m}{m} t^{s-m} y^{m-s} \cdot L_{m+n-s}^{\alpha-m+s}(x)$$

To obtain (3.3.13) we have used the following relations

$$\left[f(D_t) \cdot e^{t\phi(x,y)}\right]_{t=1} = \left[e^{t\{\phi(x,y)\}} \cdot f\{\phi(x,y)\}\right]_{t=1}$$

3.4) <u>Some other Generating Relations for $Y_n^\alpha(x;K)$ :-</u>

(A) H.M. Shrivastava [3] has given recurrence relations as follows:

3.4.1) $$\left(x\frac{\partial}{\partial x}+Kn+\alpha-x+1\right) Y_n^\alpha(x;K) = K(n+1)\, Y_{n+1}^\alpha(x;K)$$

So we consider the operator

3.4.2) $$C^* = yx\frac{\partial}{\partial x}+Ky^2\frac{\partial}{\partial y}+(\alpha-x+1)y$$

for which

3.4.3) $C^{*}\left\{y^{n}.Y_{n}^{\alpha}(x;K)\right\}=K(n+1)\,Y_{n+1}^{\alpha}(x;K)\,y^{n+1}$

The extended forms of the transformation group generated by $C^{*}$ is

3.4.4) $e^{cC^{*}}\left\{y^{n}.Y_{n}^{\alpha}(x;K)\right\}=y^{n}(1-Kcy)^{-\frac{\alpha+1}{K}-n}.$
$$.\,e^{x\left\{1-(1-Kcy)^{-1/K}\right\}}.\,Y_{n}^{\alpha}\left\{x(1-Kcy)^{-1/K};K\right\}$$

3.4.5) But

$$e^{cC^{*}}\left\{y^{n}.Y_{n}^{\alpha}(x;K)\right\}$$
$$=y^{n}\sum_{m=0}^{\infty}\binom{n+m}{m}(cKy)^{m}\,Y_{n+m}^{\alpha}(x;K)$$

Equating both values of $e^{cC^{*}}\left\{y^{n}.Y_{n}^{\alpha}(x;K)\right\}$ and making appropriate adjustments we get a generating relation as

3.4.6) $(1-t)^{-\frac{\alpha+1}{K}-n}\,.\,e^{x\left\{1-(1-t)^{-1/K}\right\}}.$
$$.\,Y_{n}^{\alpha}\left\{x(1-t)^{-1/K};K\right\}=\sum_{m=0}^{\infty}\binom{m+n}{m}t^{m}.\,Y_{n+m}^{\alpha}(x;K)$$

Special Case :- for K=1, (3.4.6) reduces to

3.4.7) $(1-t)^{-\alpha-1-n}\,e^{-\frac{xt}{1-t}}.\,L_{n}^{\alpha}\left(\frac{x}{1-t}\right)$
$$=\sum_{m=0}^{\infty}\binom{m+n}{m}t^{m}.\,L_{n+m}^{\alpha}(x)$$

(B) Following recurrence relation is also given by [3]

3.4.8) $(D-1)\,Y_{n}^{\alpha}(x;K)=-Y_{n}^{\alpha+1}(x;K)$

Let us consider an operator

3.4.9) $$C_{1}=y\frac{\partial}{\partial x}-y$$

Such that

3.4.10) $C_{1}\left\{y^{\alpha}.Y_{n}^{\alpha}(x;K)\right\}=-y^{\alpha+1}\,Y_{n}^{\alpha+1}(x;K)$

Then the extended form of the group generated by $C_1$ is

3.4.11) $e^{cC_{1}}\left\{y^{\alpha}.Y_{n}^{\alpha}(x;K)\right\}=e^{-yc}\,y^{\alpha}.\,Y_{n}^{\alpha}(x+yc;K)$

3.4.12) Also $e^{cC_1}\{y^{\alpha}, Y_n^{\alpha}(x;K)\}$

$$= \sum_{m=0}^{\infty} \frac{c^m}{\lfloor m} (-1)^m \, Y_n^{\alpha+m}(x;K) \, . \, y^{\alpha+m}$$

Equating the two values and making appropriate adjustments we get

3.4.13) $$e^{-t} \, . \, Y_n^{\alpha}(x+t;K) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\lfloor m} Y_n^{\alpha+m}(x;K)$$

where t=yc

<u>Special Case</u> :- for K =1, (3.4.13)changes to a relation for Lagguerre polynomials

3.4.14) $$e^{-t} \, . \, L_n^{\alpha}(x+t) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\lfloor m} L_n^{\alpha+m}(x)$$

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# CHAPTER IV

## Some Generating Functions for Polynomials suggested by Laguerre Polynomials - II.

4.1) **INTRODUCTION** : Konhauser [1] has considered two classes of polynomials $Y_n^\alpha(x;K)$ and $Z_n^\alpha(x;K)$ where $Y_n^\alpha(x;K)$ is a polynomials in X while $Z_n^\alpha(x;K)$ is a polynomial in $x^K$, $\alpha>-1$ and $k = 1,2,3$------. For $K = 1$, these polynomials reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$.

An explicit expression for the polynomials $Z_n^\alpha(x;K)$ was given by Konhauser in the form [1]

4.1.1) $$Z_n^\alpha(x;K) = \frac{\Gamma(Kn+\alpha+1)}{\underline{|n}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{Kj}}{\Gamma(Kj+\alpha+1)}$$

The hypergeometric representation for $Z_n^\alpha(x;K)$ is given by

4.1.2) $$Z_n^\alpha(x;K) = \frac{(\alpha+1)_{Kn}}{\underline{|n}}\ {}_1F_K\left[-n; \frac{\alpha+1}{K}, \cdots \frac{\alpha+K}{K}; \left(\frac{x}{K}\right)^K\right]$$

With the help of differential recurrence relations given in [2], we obtain differential recurrence relations for $Z_n^\alpha(x;K)$ as

4.1.3) $$\{x^{1-K} D\} Z_n^\alpha(x;K) = -K \cdot Z_{n-1}^{\alpha+K}(x;K)$$

4.1.4) $$\{x^{-\alpha} D^K (x^{\alpha+K}) - x^K\} Z_n^\alpha(x;K) = (n+1)\, Z_{n+1}^{\alpha-K}(x;K).$$

From (4.1.3) and (4.1.4) we get the following differential equations for $Z_n^\alpha(x;K)$

4.1.5) $$D^K\{x^{\alpha+1} \cdot D Z_n^\alpha(x;K)\} = x^\alpha (xD - Kn) Z_n^\alpha(x;K)$$

In the present note we shall obtain generating relations for $Z_n^{\alpha}(x;K)$ using Lie group theory.

4.2) <u>Lie operators associated with $Z_n^{\alpha}(x;K)$ :-</u>

Replacing n by $y\frac{\partial}{\partial y}$ and D by $\frac{\partial}{\partial x}$ in (4.1.5) we get the partial differential equation satisfied by $u(x,y) = y^n . Z_n^{\alpha}(x;K)$ as

4.2.1) $$Lu(x;y) = \left[\frac{\partial^K}{\partial x^K}\left(x^{\alpha+1} . \frac{\partial}{\partial x} Z_n^{\alpha}\right) - x^{\alpha}\left(x\frac{\partial}{\partial x} - Ky\frac{\partial}{\partial y}\right)Z_n^{\alpha}\right] = 0$$

Now consider the following differential operators.

4.2.2) $$A = y\frac{\partial}{\partial y}$$

$$B = y^{-1} x^{1-K}\frac{\partial}{\partial x}$$

$$C = yx^{-\alpha}\frac{\partial^K}{\partial x^K}(x^{\alpha+K}) - yx^K$$

Then

4.2.3) $$x^{-\alpha} L \equiv CB + KA$$

Also we have

4.2.4) $$A\left\{y^n . Z_n^{\alpha}(x;K)\right\} = ny^n . Z_n^{\alpha}(x;K)$$

$$B\left\{y^n . Z_n^{\alpha}(x;K)\right\} = -K . Z_{n-1}^{\alpha+K}(x;K) . y^{n-1}$$

$$C\left\{y^n . Z_n^{\alpha}(x;K)\right\} = (n+1) Z_{n+1}^{\alpha-K}(x;K) . y^{n+1}$$

The commutator relations satisfied by A, B and C are

4.2.5) $$[A,B] = -B \quad , \quad [A,C] = C$$

$$[B,C] = -K$$

These commutator relations show that $1, A, B, C$ generate a Lie group $\Gamma$

We express the extended forms of the transformation group, generated by A by solving the following

equations.

$$\frac{\partial y(p)}{\partial p} = y(p) \quad , \quad y(0) = y$$

$$\int \frac{\partial y(p)}{y(p)} = \int \partial p + K$$

$$\log y(p) = p + K$$

When $p = 0$, it gives $\log y = K$

$$\log y(p) = p + \log y$$

$$y(p) = y e^{p}$$

4.2.6) Thus $\exp pA\{f(x,y)\} = f(x, ye^{p})$

To find extended forms of the group generated by B we have to solve the following equations.

$$\frac{\partial x(b)}{\partial b} = \frac{\{x(b)\}^{1-K}}{y}$$

$$\int \frac{\partial x(b)}{\{x(b)\}^{1-K}} = \frac{1}{y}\int \partial b + R$$

$$\frac{\{x(b)\}^{K}}{K} = \frac{1}{y} b + R$$

When $b = 0$, $R = \frac{x^K}{K}$

$$\{x(b)\}^{K} = \frac{Kb}{y} + x^{K}$$

$$x(b) = \left(x^{K} + \frac{Kb}{y}\right)^{1/K}$$

Thus

4.2.7) $$\exp bB\{y^{n} f(x)\} = y^{n} e^{bB} f(x)$$

$$= y^{n} f\left\{\left(x^{K} + \frac{Kb}{y}\right)^{1/K}\right\}$$

Now to find extended forms of the group generated by $C$, the Miller's method is no longer applicable as the differential operators involved are of order $\neq 1$. for this we use well known operational formula.

$$\phi(D)\{x^{\alpha} f(x)\} = x^{\alpha} \phi\left(D + \frac{\alpha}{x}\right) f(x)$$

Now,

$$C f(x) = y x^{-\alpha} D_x^{K}\left(x^{\alpha+K} f(x)\right) - y x^{K} f(x)$$

$$= y x^{-\alpha} x^{\alpha}\left(D+\frac{\alpha}{x}\right)^{K} \cdot x^{K} f(x) - y x^{K} \cdot f(x)$$

$$= \left[y\left(D+\frac{\alpha}{x}\right)^{K} - y\right]\left\{x^{K} \cdot f(x)\right\}$$

So,

$$\mathcal{C} = y\left[\left(D+\frac{\alpha}{x}\right)^{K} - 1\right] x^{K}$$

Then we get the required result as

4.2.8) $e^{\omega \mathcal{C}}\left\{y^{n} \cdot f(x)\right\}$

$$= y^{n} \cdot e^{-\omega y x^{K}} \cdot e^{\omega y\left\{\left(D_x+\frac{\alpha}{x}\right)^{K} \cdot x^{K}\right\}} \cdot f(x)$$

Where $D_x = \dfrac{d}{dx}$

4.3) <u>Generating Functions annulled by conjugates of $(A - n)$:</u>

We see that $u(x,y) = y^{n} \cdot Z_n^{\alpha}(x;K)$ are solutions of the simultaneous equations.

$Lu = 0$ and $Au = nu$ for arbitrary $n$.

With the help of (4.2.7) and (4.2.8) we get .

4.3.1) $e^{\omega \mathcal{C}} \cdot e^{bB}\left\{y^{n} \cdot f(x)\right\}$

$$= e^{\omega \mathcal{C}} \cdot y^{n} \cdot f\left\{\left(x^{K} + \frac{Kb}{y}\right)^{1/K}\right\}$$

$$= y^{n} \cdot e^{-\omega y z^{K}} \cdot e^{\omega y\left(D+\alpha z^{-1}\right)^{K} z^{K}} \cdot f(z)$$

$$= G(x,y) \quad \text{where } z^{K} = x^{K} + \frac{Kb}{y}$$

Put $S = e^{bB + c\mathcal{C}}$ then $SAS^{-1}$ is conjugate of $A$ and then $G(x,y)$ is annulled by $L$ and $S(A-n)S^{-1}$. Now we consider the following cases :-

<u>Case I</u> : $w = 0$ , $b = 1$ , (4.3.1) reduces to

4.3.2) $e^{B}\left\{y^{n} \cdot Z_n^{\alpha}(x;K)\right\}$

$$= y^{n} \cdot Z_n^{\alpha}\left\{\left(x^{K} + \frac{K}{y}\right)^{1/K} ; K\right\}$$

Also,

$$e^{B}\{y^n . Z_n^{\alpha}(x;K)\} = \sum_{m=0}^{\infty} \frac{(B)^m}{\underline{|m}}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= \sum_{m=0}^{\infty} \frac{(B)^{m-1}}{\underline{|m}} (-K) Z_{n-1}^{\alpha+k}(x;K) . y^{n-1}$$

$$- - - -$$

$$= \sum_{m=0}^{\infty} \frac{(B)^{m-m}}{\underline{|m}} y^{n-m} (-K)^m Z_{n-m}^{\alpha+mK}(x;K)$$

Hence

4.3.3) $$e^{B}\{y^n . Z_n^{\alpha}(x;K)\} = $$

$$= y^n \sum_{m=0}^{\infty} \frac{1}{\underline{|m}} \left(-\frac{K}{y}\right)^m . Z_{n-m}^{\alpha+mK}(x;K)$$

Equating the two values and after suitable adjustments we get a generating relation as

4.3.4) $$Z_n^{\alpha}\{(x^K + Kt)^{1/K} ; K\}$$

$$= \sum_{m=0}^{\infty} K^m \frac{(-t)^m}{\underline{|m}} Z_{n-m}^{\alpha+mK}(x;K)$$

which in particular for k =1 gives

4.3.5) $$L_n^{\alpha}(x+t) = \sum_{m=0}^{\infty} \frac{(-t)^m}{\underline{|m}} L_{n-m}^{\alpha+m}(x)$$

<u>Case-II</u> : b = 0 w = 1

From (4.3.1) we get

4.3.6) $$e^{\zeta}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= y^n . e^{-yx^K} e^{y(D+\frac{\alpha}{x})^K x^K} . \{Z_n^{\alpha}(x;K)\}$$

Also,

$$e^{\zeta}\{y^n . Z_n^{\alpha}(x;K)\} = \sum_{m=0}^{\infty} \frac{(\zeta)^m}{\underline{|m}}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= \sum_{m=0}^{\infty} \frac{(\zeta)^{m-1}}{\underline{|m}} (n+1) Z_{n+1}^{\alpha-K}(x;K) . y^{n+1}$$

$$- - - -$$

$$= \sum_{m=0}^{\infty} \frac{(\zeta)^{m-m}}{\underline{|m}} \{(n+1) - - (n+m)\} Z_{n+m}^{\alpha-mK}(x;K) . y^{n+m}$$

Hence

4.3.7) $e^{\mathcal{G}}\{y^n . Z_n^{\alpha}(x;K)\}$

$$= y^n \sum_{m=0}^{\infty} (n+1)_m \frac{y^m}{\lfloor m} Z_{n+m}^{\alpha-mK}(x;K)$$

Equating the two values in after suitable adjustments we get a generating relation as

4.3.8) $$e^{y\left\{\left(D+\frac{\alpha}{x}\right)^K x^K - x^K\right\}} Z_n^{\alpha}(x;K)$$

$$= \sum_{m=0}^{\infty} (n+1)_m \frac{y^m}{\lfloor m} Z_{n+m}^{\alpha-mK}(x;K)$$

K = 1 in (4.3.8) gives

4.3.9) $$e^{y\left(D+\frac{\alpha}{x}-1\right)x} L_n^{\alpha}(x)$$

$$= \sum_{m=0}^{\infty} (n+1)_m \frac{y^m}{\lfloor m} L_{n+m}^{\alpha-m}(x)$$

which on simplifying more, turns to

4.3.10) $$e^{yDx+\alpha y - xy} L_n^{\alpha}(x)$$

$$= e^{yDx}\left(e^{\alpha y} . e^{-xy} L_n^{\alpha}(x)\right)$$

$$= e^{(\alpha+1)y} \exp(-xye^y) L_n^{\alpha}(xe^y)$$

used $e^{tDx} f(x) = e^t f(xe^t)$

$$= \sum_{m=0}^{\infty} (n+1)_m \frac{y^m}{\lfloor m} L_{n+m}^{\alpha-m}(x)$$

Case III :- $b \neq 0$ $w \neq 0$, Let $w = 1$

then (4.3.1) gives

4.3.11) $$e^{bB} e^{\mathcal{G}}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= y^n e^{-y} e^{y\left(D+\frac{\alpha}{z}\right)^K z^K} Z_n^{\alpha}(z;K)$$

where $$z^K = \left(x^K + \frac{Kb}{y}\right)$$

Also

$$e^{bB} e^{\mathcal{G}}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= e^{\mathcal{G}} e^{bB}\{y^n . Z_n^{\alpha}(x;K)\}$$

$$= e^{G} \sum_{m=0}^{\infty} y^{n-m} \frac{(-Kb)^m}{\lfloor m} Z_{n-m}^{\alpha+mK}(x;K)$$

4.3.12) $$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (n-m+1)_s \frac{y^{n-m+s}}{\lfloor m \lfloor s} (-Kb)^m \cdot Z_{n-m+s}^{\alpha+mK-sK}(x;K)$$

Equating the two values and after suitable adjustments we get a generating relation as

4.3.13) $$e^{-y} e^{y\{D+\alpha(x^K+\frac{Kb}{y})^{-1/K}\}^K} \cdot (x^K+\frac{Kb}{y}) \cdot Z_n^{\alpha}\{(x^K+\frac{Kb}{y})^{1/K};K\}$$

$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (-Kb)^m (n-m+1)_s \frac{y^{s-m}}{\lfloor s \lfloor m} Z_{n-m+s}^{\alpha+mK-sK}(x;K)$$

In particular for $k = 1$, (4.3.13) gives

4.3.14) $$e^{-y} e^{y(D+\alpha z^{-1})z} \{L_n^{\alpha}(z)\}$$

Where $z = x + \frac{b}{y}$

$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (-b)^m (n-m+1)_s \frac{y^{s-m}}{\lfloor s \lfloor m} L_{n-m+s}^{\alpha+m-s}(x)$$

which on further simplifications gives

$$e^{-y} e^{yDz} \{e^{\alpha y z^2 - yz} L_n^{\alpha}(z)\}$$

$$= e^{-y} e^{y} \exp\{\alpha y z^2 e^{2y} - y z e^{y}\} L_n^{\alpha}(z e^{y})$$

$$= \exp\{\alpha y e^{2y}(x+\frac{b}{y})^2 - y e^{y}(x+\frac{b}{y})\} \cdot$$

4.3.15) $$\cdot L_n^{\alpha}\{e^{y}(x+\frac{b}{y})\}$$

$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (-b)^m (n-m+1)_s \frac{y^{s-m}}{\lfloor s \lfloor m} L_{n-m+s}^{\alpha+m-s}(x)$$

4.4) <u>Some more generating Relations for</u> $Z_n^{\alpha}(x;K)$ :-

(A) H.M. Shrivastava [2] has given recurrence relations for $Z_n^{\alpha}(x;K)$

4.4.1) $$\left(x\frac{\partial}{\partial x}+\alpha\right) Z_n^{\alpha}(x;K) = (Kn+\alpha) Z_n^{\alpha-1}(x;K)$$

So we consider the operator

4.4.2) $B_1 = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\alpha}{y}$

Then we observe that

4.4.3) $B_1\{y^\alpha . Z_n^\alpha(x;K)\} = (Kn+\alpha)\, Z_n^{\alpha-1}(x;K) . y^{\alpha-1}$

Using the same method used by Miller [3] we get the extended form of the transformation group generated by $B_1$ as

4.4.4) $e^{bB_1}\{y^\alpha . Z_n^\alpha(x;K)\}$

$$= y^\alpha . e^{\alpha b/y}\, Z_n^\alpha(x e^{b/y};K)$$

4.4.5) Also $e^{bB_1}\{y^\alpha . Z_n^\alpha(x;K)\}$

$$= y^\alpha \sum_{m=0}^{\infty} (Kn+\alpha-m+1)_m \left(\frac{b}{y}\right)^m \frac{1}{\lfloor m}\, Z_n^{\alpha-m}(x;K)$$

equating both values of $e^{bB_1}\{y^\alpha . Z_n^\alpha(x;K)\}$ and making appropriate adjustments we get

4.4.6) $e^{\alpha t}\, Z_n^\alpha(xe^t;K)$

$$= \sum_{m=0}^{\infty} (Kn+\alpha-m+1)_m \frac{t^m}{\lfloor m}\, Z_n^{\alpha-m}(x;K)$$

Where $t = \frac{b}{y}$

Which in particular for $K = 1$ gives

4.4.7) $e^{\alpha t} . L_n^\alpha(xe^t)$

$$= \sum_{m=0}^{\infty} (n+\alpha-m+1)_m \frac{t^m}{\lfloor m}\, L_n^{\alpha-m}(x)$$

as $Z_n^\alpha(x;1) = L_n^\alpha(x)$

(B) Following recurrence relations is also given by [2] for $Z_n^\alpha(x;K)$

4.4.8) $(Kn - xD)\, Z_n^\alpha(x;K) = K(Kn+\alpha-K+1)_K\, Z_{n-1}^\alpha(x;K)$

So let us consider an operator

4.4.9) $$B_2 = K\frac{\partial}{\partial y} - \frac{x}{y}\frac{\partial}{\partial x}$$

Then

4.4.10) $$B_2\{y^n . Z_n^\alpha(x;K)\}$$
$$= K(Kn+\alpha-K+1)_K \cdot y^{n-1} Z_{n-1}^\alpha(x;K)$$

The extended form of the transformation group generated by $B_2$ is

4.4.11) $$e^{bB_2}\{y^n . Z_n^\alpha(x;K)\}$$
$$= (Kb+y)^n . Z_n^\alpha\left\{\frac{x\,y^{1/K}}{(Kb+y)^{1/K}};K\right\}$$

It has been obtained by using the method of Miller [3] Also

4.4.12) $$e^{bB_2}\{y^n . Z_n^\alpha(x;K)\}$$
$$= \sum_{m=0}^{\infty}\frac{b^m}{\lfloor m}B_2^m\{y^n . Z_n^\alpha(x;K)\}$$
$$= y^n\sum_{m=0}^{\infty}\frac{1}{\lfloor m}\left(\frac{Kb}{y}\right)^m Z_{n-m}^\alpha(x;K) \cdot$$
$$\cdot (Kn+\alpha-mK+1)_{mK}$$

Hence Equating the two values and making appropriate adjustments we get

4.4.13) $$(1+t)^n Z_n^\alpha\left\{\frac{x}{(1+t)^{1/K}};K\right\}$$
$$= \sum_{m=0}^{\infty}\frac{t^m}{\lfloor m}(Kn+\alpha-mK+1)_{mK} \cdot Z_{n-m}^\alpha(x;K)$$

Which in particular for $K = 1$ gives

4.4.14) $$(1+t)^n L_n^\alpha\left(\frac{x}{1+t}\right)$$
$$= \sum_{m=0}^{\infty}\frac{t^m}{\lfloor m}(n+\alpha-m+1)_m L_{n-m}^\alpha(x)$$

as $Z_n^\alpha(x;K) = L_n^\alpha(x)$

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# CHAPTER V

## LIE THEORY AND HYPERGEOMETRIC FUNCTIONS $_2F_1$ :-

5.1. INTRODUCTION : Louis Weisner has introduced in [4] to find Lie Operators and then generating functions of $_2F_1$ (a, b, c, z) corresponding to raising and lowering of parameter a. Manocha [5] B.M. Agarwal & Renu Jain [6] etc. have tried to obtain generating functions by variation of parameter a. In the present paper, the Authors have tried to find Lie Operators effecting the parameter c and obtained the generating functions corresponding to lowering and raising of parameter c.

For the reasons of convergence, - throughout this chapter, we take $|z|<1$

5.2. Differential Equation for $_2F_1(a,b;c+n;z)$ :-

Sla ter [3] has given following two differential recurrence relations for $_2F_1(a,b;c+n;z)$

5.2.1) $$\left[z\frac{\partial}{\partial z}+c+n-1\right] {}_2F_1(a,b;c+n;z) = (c+n-1)\ {}_2F_1(a,b;c+n-1;z)$$

5.2.2) $$\left[(c+n)(1-z)\frac{\partial}{\partial z}-(c+n)(a+b-c-n)\right]\cdot {}_2F_1(a,b;c+n;z) = (c+n-a)(c+n-b)\,.\,{}_2F_1(a,b;c+n+1;z)$$

From (5.2.1) and (5.2.2) we get the following differential equation for $_2F_1(a,b;c+n;z)$

$$(c+n)\left[\left(z\frac{\partial}{\partial z}+n+c-1\right)\left\{(1-z)\frac{\partial}{\partial z}-(a+b-c-n)\right\} - (c+n-a)(c+n-b)\right]u=0$$

5.2.3) Now replace $n$ by $y\frac{\partial}{\partial y}$ we get

the partial differential equation satisfied by $y^n$.
${}_2F_1(a,b,c+n;z)$ as

5.2.4) $$\left(c+y\frac{\partial}{\partial y}\right)\Big[\left(z\frac{\partial}{\partial z}+y\frac{\partial}{\partial y}+c-1\right)\cdot\Big\{(1-z)\frac{\partial}{\partial z}-a-b+c+y\frac{\partial}{\partial y}\Big\}-\left(c+y\frac{\partial}{\partial y}-a\right)\left(c+y\frac{\partial}{\partial y}-b\right)\Big]u=0$$

Now consider the following differential operators

5.2.5) $$A = y\frac{\partial}{\partial y}$$

$$B = \frac{z}{y}\frac{\partial}{\partial z}+\frac{\partial}{\partial y}+\frac{c-1}{y}$$

$$\mathcal{C} = (1-z)cy\frac{\partial}{\partial z}+(1-z)y^2\frac{\partial^2}{\partial y\partial z}+y^3\frac{\partial^2}{\partial y^2}+(2c-a-b+1)y^2\frac{\partial}{\partial y}-yc(a+b-c)$$

Then

5.2.6) $$(c+A)L \equiv B\mathcal{C}-(A+\mathcal{C}-a)(A+\mathcal{C}-b)\cdot(A+\mathcal{C})$$

Also

5.2.7) $$A\{y^n\cdot{}_2F_1(a;b;c+n;z)\} = ny^n\cdot{}_2F_1(a,b;c+n;z)$$

$$B\{y^n\cdot{}_2F_1(a,b;c+n;z)\} = (c+n-1)y^{n-1}\cdot{}_2F_1(a,b;c+n-1;z)$$

$$\mathcal{C}\{y^n\cdot{}_2F_1(a,b;c+n;z)\} = (c+n-a)(c+n-b)y^{n+1}\cdot{}_2F_1(a,b;c+n+1;z)$$

The commutator relations satisfied by $A$, $B$ and $\mathcal{C}$ are

5.2.8) $$[A,B]=-B, \qquad [A,\mathcal{C}]=\mathcal{C}$$

$$[B,\mathcal{C}] = \{(3c^2+ac^2-3c-2ac-2bc+a+1+ab+b)+(bc-3-2a-2b)A+3A^2\}$$

These commutator relations show that 1, $A$, $B$, $\mathcal{C}$ generate a Lie group.

5.2.9) We put

$$\mathcal{B} = \frac{z}{y}\frac{\partial}{\partial z} + \frac{c+n-1}{y}$$

$$\mathcal{C} = y(c+n)(1-z)\frac{\partial}{\partial z} - y(c+n)(a+b-c-n)$$

(i) Now to find extended forms of the group generated by $A$ we have to solve as follows :

$$\frac{\partial y(a)}{\partial a} = y(a) \quad , \quad y(0) = y \, .$$

$$\int \frac{\partial y(a)}{y(a)} = \int \partial a + K$$

$$\log y(a) = a + K$$

When $a = 0$, it gives $\operatorname{Log} y = K$

$$\log y(a) = a + \log y$$

$$y(a) = y e^{a}$$

5.2.10) $$\exp aA\{f(y,z)\} = f(ye^{a}, z)$$

(ii) To find $\exp w\mathcal{B}$ We have to solve following equations as directed in Miller [1.]

$$\frac{dz(\omega)}{d\omega} = \frac{z(\omega)}{y} \quad \text{and} \quad \frac{dv(\omega)}{d\omega} = \frac{c+n-1}{y} v(\omega)$$

$$\frac{dz(\omega)}{z(\omega)} = \frac{1}{y} d\omega \qquad \frac{dv(\omega)}{v(\omega)} = \frac{c+n-1}{y} d\omega$$

$$\log z(\omega) = \frac{\omega}{y} + K \qquad \log v(\omega) = \frac{c+n-1}{y}\omega + K$$

when $w = 0$, $z(0) = z$ when $\omega = 0$, $v(0) = 1$

then $K = \log z$ then $K = 0$

$$\log z(\omega) = \frac{\omega}{y} + \log z \qquad v(\omega) = e^{(c+n-1)\omega/y}$$

$$z(\omega) = z e^{\omega/y}$$

5.2.11) $$\exp \omega \mathcal{B}\{y^{p} f(z)\} = y^{p} e^{(c+n-1)\omega/y} f(ze^{\omega/y})$$

(iii) To find $\exp \mu \mathcal{C}$ we have to integrate as follows:

$$\frac{dz(\mu)}{d\mu} = y(c+n)\{1-z(\mu)\}$$

$$\int \frac{dz(\mu)}{1-z(\mu)} = y(c+n)\int d\mu + K$$

$$-\log\{1-z(\mu)\} = (c+n)y\mu + K$$

when $\mu = 0$, $z(0) = z$, $K = -\log(1-z)$

$$-\log\{1-z(\mu)\} = (c+n)y\mu - \log(1-z)$$

$$\frac{1-z}{1-z(\mu)} = e^{(c+n)y\mu}$$

So,

$$z(\mu) = 1-(1-z)e^{-(c+n)\mu y}$$

and,

$$\frac{dv(\mu)}{d\mu} = -y(c+n)(a+b-c-n).v(\mu)$$

$$\log v(\mu) = -y(c+n)(a+b-c-n)\mu + K$$

when

$\mu = 0$, $v(0) = 1$, $K = 0$

So,

$$\log v = -y(c+n)(a+b-c-n)\mu$$

$$v = e^{(c+n)(c+n-a-b)\mu y}$$

5.2.12) $$\exp \mu G\{y^{p}.f(z)\} = y^{p}e^{\mu y(c+n)(c+n-a-b)} \cdot f\{1-(1-z)e^{-(c+n)\mu y}\}$$

5.3.) <u>Generating Functions annulled by Conjugates of $(A - n)$</u>:- We see that $u = y^{n}{}_{2}F_{1}(a, b; c+n; z)$ are solutions of the simultaneous equations $Lu = 0$ and $Au = nu$ for arbitrary n. Now.

5.3.1) $$e^{\mu G}.e^{\omega B}\{y^{n}.{}_{2}F_{1}(a,b;c+n;z)\}$$
$$= y^{n}e^{(c+n-1)\frac{\omega}{y} + \mu y(c+n)(c+n-a-b)} \cdot {}_{2}F_{1}\left[a,b;c+n;\{1-(1-ze^{\omega/y})e^{-(c+n)\mu y}\}\right]$$
$$= G(x,y)$$

Put $S = e^{\omega B + \mu G}$ then $SAS^{-1}$ is a conjugate of $A$ and $G(x,y)$ is annulled by $L$ and

$S(A-n)S^{-1}$

Now we consider the following cases :-

<u>Case - I</u> : $w = 0$, $\mu = 1$

Then ( 5.3.1) reduces to

5.3.2) $$e^{\mathcal{G}}\{y^n \cdot {}_2F_1(a,b;c+n;z)\}$$
$$= y^n \cdot e^{y(c+n)(c+n-a-b)} \cdot$$
$$\cdot {}_2F_1\left[a,b;c+n;\{1-(1-z)\,e^{-(c+n)y}\}\right]$$

Also,

$$e^{\mathcal{G}}\{y^n \cdot {}_2F_1(a,b;c+n;z)\}$$
$$= \sum_{m=0}^{\infty} \frac{(\mathcal{G})^m}{\lfloor m}\left[y^n \cdot {}_2F_1(a,b;c+n;z)\right]$$
$$= \sum_{m=0}^{\infty} \frac{(\mathcal{G})^{m-1}}{\lfloor m}(c+n-a)(c+n-b)\,y^{n+1} \cdot$$
$$\cdot {}_2F_1(a,b;c+n+1;z)$$

$$= \sum_{m=0}^{\infty} \frac{(\mathcal{G})^{m-m}}{\lfloor m}\{(c+n-a)\cdots(c+n-a+m-1) \cdot$$
$$\cdot (c+n-b)\cdots(c+n-b+m-1) \cdot$$
$$\cdot y^{n+m} \cdot {}_2F_1(a,b;c+n+m;z)$$
$$= \sum_{m=0}^{\infty} \frac{1}{\lfloor m}(c+n-a)_m\,(c+n-b)_m\,y^{n+m} \cdot$$
$$\cdot {}_2F_1(a,b;c+n+m;z)$$

5.3.3) So, $$e^{\mathcal{G}}\{y^n \cdot {}_2F_1(a,b;c+n;z)\}$$
$$= \sum_{m=0}^{\infty} \frac{1}{\lfloor m}(c+n-a)_m\,(c+n-b)_m\,y^{n+m} \cdot {}_2F_1(a,b;c+n+m;z)$$

Equating the two values and after minor adjustments we get

5.3.4) $$e^{y(c+n)(c+n-a-b)} \cdot {}_2F_1\left[a,b;c+n;\{1-(1-z) \cdot\right.$$
$$\left.\cdot\, e^{-(c+n)y}\}\right]$$

$$= \sum_{m=0}^{\infty} (c+n-a)_m (c+n-b)_m \frac{y^m}{\lfloor m} \cdot {}_2F_1(a,b;c+n+m;z)$$

where $|z|<1$, $|1-(1-z)e^{-(c+n)y}|<1$

Case II $\quad w = 1 \quad , \quad \mu = 0$

Then (5.3.1) reduces to

5.3.5) $$e^{B}\{y^n \cdot {}_2F_1(a,b;c+n;z)\} = y^n e^{\frac{c+n-1}{y}} \cdot {}_2F_1[a,b;c+n;ze^{1/y}]$$

Also,

$$e^{B}\{y^n \cdot {}_2F_1(a,b;c+n;z)\}$$
$$= \sum_{m=0}^{\infty} \frac{(B)^m}{\lfloor m} [y^n \cdot {}_2F_1\{a,b;c+n;z\}]$$
$$= \sum_{m=0}^{\infty} \frac{(B)^{m-1}}{\lfloor m} (c+n-1)\, y^{n-1} \cdot {}_2F_1\{a,b;c+n-1;z\}$$

$$- - - -$$

$$= \sum_{m=0}^{\infty} \frac{(B)^{m-m}}{\lfloor m} (c+n-1) \cdots (c+n-m)\, y^{n-m} \cdot {}_2F_1[a,b;c+n-m;z]$$

So,

5.3.6) $$e^{B}\{y^n \cdot {}_2F_1(a,b;c+n;z)\} = \sum_{m=0}^{\infty} \frac{(c+n-m)_m}{\lfloor m} y^{n-m} \cdot {}_2F_1[a,b;c+n-m;z]$$

Equating the two values and after appropriate adjustments we get

5.3.7) $$e^{\frac{c+n-1}{y}} \cdot {}_2F_1[a,b;c+n;ze^{1/y}] = \sum_{m=0}^{\infty} \frac{(c+n-m)_m}{y^m \lfloor m} \cdot {}_2F_1[a,b;c+n-m;z]$$

$|z|<1$ and $|ze^{1/y}|<1$

Special case : For $n = 0$

5.3.8) $$e^{\frac{c-1}{y}} \cdot {}_2F_1[a,b;c;ze^{1/y}]$$

$$= \sum_{m=0}^{\infty} \frac{(c-m)_m}{y^m \lfloor m} \cdot {}_2F_1 [a, b; c-m; z]$$

Case III :- $w \neq 0$ , $\mu = 1$

Then (5.3.1) reduces to

5.3.9) $$e^{G} . e^{\omega B} \{ y^n . {}_2F_1 (a, b; c+n; z) \}$$
$$= y^n . e^{(c+n-1)\frac{\omega}{y} + y(c+n)(c+n-a-b)} .$$
$$\cdot {}_2F_1 \left[ a, b; c+n; \{ 1-(1-ze^{\omega/y}) e^{-(c+n)y} \} \right]$$

Also,

$$e^{G} e^{\omega B} \{ y^n . {}_2F_1 (a, b; c+n; z) \}$$
$$= e^{G} \sum_{m=0}^{\infty} \frac{\omega^m}{\lfloor m} (c+n-m)_m y^{n-m} .$$
5.3.10) $$\cdot {}_2F_1 (a, b; c+n-m; z)$$
$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega^m}{\lfloor m \lfloor s} (c+n-m)_m \cdot$$
$$\cdot (c+n-m-a)_s (c+n-m-b)_s y^{n-m+s}$$
$$\cdot {}_2F_1 (a, b; c+n-m+s; z)$$

Equating both values and after appropriate adjustments we get

5.3.11) $$e^{(c+n-1)\frac{\omega}{y} + y(c+n)(c+n-a-b)} .$$
$$\cdot {}_2F_1 \left[ a, b; c+n; \{ 1-(1-ze^{\omega/y}) e^{-(c+n)y} \} \right]$$
$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega^m}{\lfloor m \lfloor s} (c+n-m)_m (c+n-m-a)_s \cdot$$
$$\cdot (c+n-m-b)_s y^{s-m} . {}_2F_1 (a, b; c+n-m+s; z)$$

where $|z| < 1$ and $| 1-(1-ze^{\omega/y}) e^{-(c+n)y} | < 1$

Special Case :- for n = 0

5.3.12) $$e^{(c-1)\frac{\omega}{y} + yc(c-a-b)}$$
$${}_2F_1 \left[ a, b; c; \{ 1-(1-ze^{\omega/y}) e^{-cy} \} \right]$$

$$= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega^m}{\lfloor m \lfloor s} (c-m)_m (c-m-a)_s (c-m-b)_s \cdot$$

$$\cdot y^{s-m} \; {}_2F_1(a, b; c-m+s; z)$$

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# CHAPTER VI

## ON CERTAIN GENERATING RELATIONS INVOLVING CLASSICAL POLYNOMIALS.

1. INTRODUCTION : Shrivastava and Singh (1) in an attempt to unify the classical polynomials, considered a class of functions defined by relation -

6.1.1) $$P_n^{(\alpha,\beta,K)}(x,r,s,m,A,B) = (Ax+B)^{-\alpha}(1-Kx^r)^{-\beta/K} \cdot D^n\left[(Ax+B)^{\alpha+mn}(1-Kx^r)^{\beta/K+sn}\right]$$

where $\alpha$, $\beta$ k, s, m, A and B are all parameters with suitable restrictions as per requirement. The above generalisation includes the classical polynomials and functions of Mathematical Physics as special cases. In particular we mention few below :-

Jacobi Polynomials :

6.1.2) $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{\lfloor n}P_n^{(\alpha,\beta,1)}(x,1,1,1,1,1)$$
$$= \frac{(-1)^n}{\lfloor n}P_n^{(\beta,-\alpha,-1)}(x,1,1,1,-1,1)$$
$$= \frac{(-1)^n}{\lfloor n}P_n^{(\alpha,\beta,1)}\left(\frac{1+x}{2},1,1,1,1,0\right)$$
$$= \frac{1}{\lfloor n}P_n^{(\alpha,\beta,-1)}\left(\frac{1-x}{2},1,1,1,1,0\right)$$

Laguerre Polynomials :-

6.1.3) $$L_n^{(\alpha)}(x) = \lim_{K\to 0}\frac{1}{\lfloor n}P_n^{(\alpha,1,K)}(x,1,0,1,1,0)$$

Hermite Polynomials :-

6.1.4) $$H_n(x) = \lim_{K\to 0}P_n^{(0,2,K)}(x,1,0,0,1,0)$$
$$= \lim_{K\to 0}(-1)^n P_n^{(0,1,K)}(x,2,0,0,1,0)$$

Bessel Polynomials :

6.1.5) $y_n(x, a+2, b) = \lim_{K\to 0} b^{-n} P_n^{(a,b,K)}(x, -1, 0, 2, 1, 0)$

Other than above, we may mentiond the names of Gagenbauer polynomials, Generalised Functions of Gould Hopper [2] Generalised Laguerre Functions of Singh - Shrivastava [3] and Chatterjea [4] etc. which are all included in [1].

In the present note, we shall derive Mehler's type formulas for the function (1.1) by group theoretic approach when a bilinear generating relation is known.

6.2) Lie - Operators for $P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)$ :-

Shrivastava - Singh [1] has given the operational relation

6.2.1) $$\left(D + \frac{\alpha A}{Ax+B} + \frac{\beta r x^{r-1}}{1-Kx^r}\right) P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)$$
$$= (Ax+B)^{-m}(1-Kx^r)^{-s} P_{n+1}^{(\alpha-m,\beta-Ks,K)}(x, r, s, m, A, B)$$

Now put

6.2.2) $$\Lambda_{y,x} = y(Ax+B)^{m-1}(1-Kx^r)^{s-1}\cdot\left[(Ax+B)(1-Kx^r)\frac{\partial}{\partial x} + \alpha A(1-Kx^r) + \beta r x^{r-1}(Ax+B)\right]$$

So that

6.2.3) $$\Lambda_{y,x}\left\{P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)\, y^n\right\}$$
$$= P_{n+1}^{(\alpha-m,\beta-Ks,K)}(x, r, s, m, A, B)\, y^{n+1}$$

Hence clearly $\Lambda_{y,x}$ forms a raising ~~the~~ Lie operator for class of functions $P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)$

The extended form of this operator is given by

6.2.4) $$e^{\omega \Lambda_{y,x}} f(x,y) = (Ax+B)^{-\alpha}(1-Kx^r)^{-\beta/K} \cdot$$

$$\cdot \left[A(x+t)+B\right]^{\alpha}\left[1-K(x+t)^r\right]^{\beta/K} \cdot$$

$$\cdot f(x+t, y)$$

where $t = \omega y (Ax+B)^m (1-Kx^r)^s$

(6.2.4) has been obtained by using the following generating relation in [1]

6.2.5) $$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} P_n^{(\alpha-mn,\ \beta-Ksn,\ K)}(x, r, s, m, A, B)$$

$$= (Ax+B)^{-\alpha}(1-Kx^r)^{-\beta/K} \cdot$$

$$\cdot \left[A\left\{x+t(Ax+B)^m(1-Kx^r)^s\right\}+B\right]^{\alpha}$$

$$\cdot \left[1-K\left\{x+t(Ax+B)^m(1-Kx^r)^s\right\}^r\right]^{\beta/K}$$

6.3) <u>Mehler's type formulas</u> :

Consider the bilinear generating function, supposed to exist for $P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)$ as

6.3.1) $$G(x, z, \omega) = \sum_{n=0}^{\infty} a_n \omega^n P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B) \cdot$$

$$\cdot P_n^{(r,\delta,\lambda)}(z, p, q, l, C, D)$$

Putting $\omega = \omega t_1 t_2 g$ we get

6.3.2) $$G(x, z, \omega t_1 t_2 g) = \sum_{n=0}^{\infty} a_n \left\{t_1^n P_n^{(\alpha,\beta,K)}(x, r, s, m, A, B)\right\} \cdot$$

$$\cdot \left\{P_n^{(r,\delta,\lambda)}(z, p, q, l, C, D)\, t_2^n\right\}(\omega g)^n$$

Now operating both sides by

$$e^{\omega \Lambda_{x,t_1}} \quad ; \quad e^{\omega \Lambda_{z,t_2}}$$

with appropriate adjustments of parameter, we get by (6.2.3)

6.3.3) $$\left(\exp \omega \Lambda_{x,t_1}\right)\left(\exp \omega \Lambda_{x,t_2}\right) \cdot$$

$$\cdot G(x, z, \omega t_1 t_2 g)$$

$$= \sum_{n=0}^{\infty} a_n \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{\omega^{j_1}}{\lfloor j_1} t_1^{j_1+n} P_{j_1+n}^{(\alpha-mj_1,\beta-j_1Ks,K)}(x,r,s,m,A,B).$$

$$\cdot \frac{\omega^{j_2}}{\lfloor j_2} t_2^{n+j_2} P_{j_2+n}^{(\nu-\ell j_2,\delta-\lambda j_2,\lambda)}(z,p,q,\ell,C,D)\ (\omega g)^n$$

But by (6.2.4)

6.3.4) $$(\exp \Lambda_{x,t_1})(\exp \Lambda_{z,t_2})G(x,z,\omega t_1t_2g)$$

$$= (Ax+B)^{-\alpha}(1-Kx^r)^{-\beta/K}$$

$$\cdot \left[A\{x+\omega t_1(Ax+B)^m(1-Kx^r)^s\}+B\right]^{\alpha}$$

$$\cdot \left[1-K\{x+\omega t_1(Ax+B)^m(1-Kx^r)^s\}^r\right]^{\beta/K}$$

$$\cdot (CZ+D)^{-\nu}(1-\lambda z^p)^{-\delta/\lambda}$$

$$\cdot \left[C\{z+t_2\omega(CZ+D)^{\ell}(1-\lambda z^p)^q\}+D\right]^{\nu}$$

$$\cdot \left[1-\lambda\{z+t_2\omega(CZ+D)^{\ell}(1-\lambda z^p)^q\}^p\right]^{\delta/\lambda}$$

$$\cdot G\left[x+t_1\omega(Ax+B)^m(1-Kx^r)^s,\ z+t_2\omega(CZ+D)^{\ell}(1-\lambda z^p)^q,\ \omega t_1t_2g\right]$$

From (6.3.3) and (6.3.4), on putting $t_1=t_2=1$ and after series manupulation we get

6.3.5) $$(Ax+B)^{-\alpha}(1-Kx^r)^{-\beta/K}$$

$$\cdot \left[A\{x+\omega(Ax+B)^m(1-Kx^r)^s\}+B\right]^{\alpha}$$

$$\cdot \left[1-K\{x+\omega(Ax+B)^m(1-Kx^r)^s\}^r\right]^{\beta/K}$$

$$\cdot (Cz+D)^{-\nu}(1-\lambda z^p)^{-\delta/\lambda}$$

$$\cdot \left[C\{z+\omega(Cz+D)^{\ell}(1-\lambda z^p)^q\}+D\right]^{\nu}$$

$$\cdot \left[1-\lambda\{z+\omega(Cz+D)^{\ell}(1-\lambda z^p)^q\}^p\right]^{\delta/\lambda}$$

$$\cdot G\left[x+\omega(Ax+B)^m(1-Kx^r)^s,\ z+\omega(Cz+D)^{\ell}(1-\lambda z^p)^q,\ \omega g\right]$$

$$= \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\min(n,j_1)} \frac{a_{n-j_2}}{\underline{|j_1-j_2}\ \underline{|j_2}}\, P_{j_1+n-2j_2}^{(\alpha-mj_1+mj_2,\ \beta-(j_1-j_2)KS)}(x, r, s, m, A, B)$$

$$\cdot\, P_n^{(\gamma-lj_2,\ \delta-j_2\lambda q,\ \lambda)}(z, p, q, l, C, D)\, .\, w^{n+j_1-j_2}\, g^{n-j_2}$$

Thus we have proved the theorem :-

If there exists a bilinear generating function of the form ( 6.3.1), then there exists a generating relation of the form (6.3.5).

6.4) <u>Applications of the Theorem:-</u>

Although the above theorem can be applied to many well known classical polynomials, we given below two special cases for Jacobi polynomials and Hermite polynomials respectively.

(A) <u>Jacobi Polynomials</u>:- Manocha [5] has given following bilinear generating function for Jacobi Polynomials. $P_n^{(\alpha,\beta)}(x)$ as

6.4.1) $$\sum_{n=0}^{\infty} \frac{(\lambda)_n\ \underline{|n}}{(-\alpha-\beta)_n\,(-\gamma-\delta)_n} P_n^{(\alpha-n,\beta-n)}(x)\cdot P_n^{(\gamma-n,\delta-n)}(y)\, t^n$$

$$= \left[1-\tfrac{1}{4}(1+x)(1+y)t\right]^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(-\beta)_n\,(-\delta)_n}{\underline{|n}\ (-\alpha-\beta)_n\,(-\gamma-\delta)_n}$$

$$\cdot\, \frac{t^n}{\left[1-\frac{1}{4}(1+x)(1+y)t\right]^n}$$

$$\cdot\, F_2\left[\lambda+n, \beta+n, -\delta+n, -\alpha-\beta+n, -\gamma-\delta+n, \frac{-(1+y)t}{2\left\{1-\frac{1}{4}(1+x)(1+y)t\right\}}, \frac{-(1+x)t}{2\left\{1-\frac{1}{4}(1+x)(1+y)t\right\}}\right]$$

We know the Rodrigues form of Jacobi Polynomials

$$P_n^{(\alpha-n,\beta-n)}(x) = \frac{(-1)^n}{2^n\,\underline{|n}}(1-x)^{-\alpha+n}(1+x)^{-\beta+n}\,.\,D^n\left[(1-x)^{\alpha}(1+x)^{\beta}\right]$$

On replacing x by -x we get

$$P_n^{(\alpha-n,\beta-n)}(-x) = \frac{(-1)^n}{2^n\,\underline{|n}}(1+x)^{-\alpha+n}(1-x)^{-\beta+n}\,.$$

$$\cdot D^n\left[(1+x)^{\alpha}(1-x)^{\beta}\right]$$

With the help of ( 6.1.1) we get

$$\frac{P_n^{(\alpha-n,\beta-n)}(-x)}{(1-x^2)^n}=\frac{(-1)^n}{2^n\,\underline{|n}}\,P_n^{(\alpha,\beta,1)}(x,1,0,0,1,1)$$

Put the expression in L.H.S of ( 6.4.1) which gives

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n\,\underline{|n}}{(-\alpha-\beta)_n\,(-\nu-\delta)_n}\,\frac{(-1)^{2n}\,(1-x^2)^{2n}}{2^{2n}\,\underline{|n}\,\underline{|n}}\,.$$

$$\cdot P_n^{(\alpha,\beta,1)}(x,1,0,0,1,1)\,.\,P_n^{(\nu,\delta,1)}(z,1,0,0,1,1)t^n$$

which is of the form (6.3.1), Hence by the theorem proved the form ( 6.3.5) should exist

6.4.2)
$$(1+x)^{-\alpha}(1-x)^{-\beta}\left[1+x+\frac{t(1-x^2)^2}{4}\right]^{\alpha}.$$

$$\cdot\left[1-x-\frac{t(1-x^2)^2}{4}\right]^{\beta}(1+z)^{-\nu}(1-z)^{-\delta}$$

$$\cdot\left[1+z+\frac{t(1-x^2)^2}{4}\right]^{\nu}\left[1-z-\frac{t(1-x^2)^2}{4}\right]^{\delta}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(-\beta)_n\,(-\delta)_n}{(-\alpha-\beta)_n\,(-\nu-\delta)_n}\cdot\frac{g^n t^n(1-x^2)^{2n}}{2^{2n}}$$

$$\cdot\left[1-\frac{1}{4}\left\{1-x-\frac{t(1-x^2)^2}{4}\right\}\left\{1-z-\frac{t(1-x^2)^2}{4}\right\}.\right.$$

$$\left.\cdot\frac{gt(1-x^2)^2}{4}\right]^{-n-1}$$

$$\cdot F_2\left[1+n,\beta+n,-\delta+n,-\alpha-\beta+n,-\nu-\delta+n,\right.$$

$$,\frac{-\left\{1-z-\frac{t(1-x^2)^2}{4}\right\}\frac{gt(1-x^2)^2}{4}}{2\left[1-\frac{1}{4}\left\{1-x-\frac{t(1-x^2)^2}{4}\right\}\left\{1-z-\frac{t(1-x^2)^2}{4}\right\}\frac{gt(1-x^2)^2}{4}\right]}$$

$$,\frac{-\left\{1-x-\frac{t(1-x^2)^2}{4}\right\}\frac{gt(1-x^2)^2}{4}}{2\left[1-\frac{1}{4}\left\{1-x-\frac{t(1-x^2)^2}{4}\right\}\left\{1-z-\frac{t(1-x^2)^2}{4}\right\}\frac{gt(1-x^2)^2}{4}\right]}$$

$$= \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\min(n,j_1)} (\lambda)_{n-j_2} \left(\frac{x^2-1}{2}\right)^{j_1} \frac{\lfloor n \; \lfloor n+j_1-2j_2}{(-\alpha-\beta)_{n-j_2}(-\gamma-\delta)_{n-j_2}} .$$

$$\cdot \frac{t^{n+j_1-j_2} \; g^{n-j_2}}{\lfloor n-j_2 \; \lfloor j_1-j_2 \; \lfloor j_2} \; P_n^{(\alpha-n,\,\beta-n)}(z).$$

$$\cdot \; P_{n+j_1-2j_2}^{(\alpha-n-j_1+2j_2,\;\beta-n-j_1+2j_2)}(x)$$

(B) <u>Hermite Polynomials</u> :- Carlitz [6] has given following bilinear generating function for Hermite Polynomials. $H_n(x)$ as

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y) \frac{t^n}{2^n \lfloor n}$$

6.4.3)
$$= (1-t^2)^{-1/2} \exp \frac{2xyt - (x^2+y^2)t^2}{1-t^2}$$

We know the Rodrigues form of Hermite Polynomials

$$H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2})$$

With the help of (6.1.4) we get

$$H_n(x) = \lim_{K\to 0} (-1)^n P_n^{(0,2,K)}(x,1,0,0;1,0)$$

Put this expression in L.H.S of (6.4.3) which gives

$$\lim_{K\to 0} \sum_{n=0}^{\infty} P_n^{(0,2,K)}(x,1,0,0;1,0)\, P_n^{(0,2,\lambda)}(y,1,0,0;1,0) \frac{t^n}{2^n \lfloor n}$$

which is of the form(6.3.1), Hence by the Theorem proved the form ( 6.3.5) should exist

6.4.4)
$$(1-t^2g^2)^{-1/2}\, e^{\{x^2-(x+t)^2\}} \cdot e^{\{z^2-(z+t)^2\}}$$

$$\cdot \exp \frac{2(x+t)(z+t)tg - \{(x+t)^2+(z+t)^2\}t^2g^2}{1-t^2g^2}$$

$$= \sum_{n=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\min(n,j_1)} \frac{(-1)^{j_1}\, t^{n+j_1-j_2}\, g^{n-j_2}}{2^{n-j_2} \lfloor n-j_2 \; \lfloor j_1-j_2 \; \lfloor j_2} .$$

$$\cdot \; H_{n+j_1-2j_2}(x) \quad H_n(z)$$

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# CHAPTER VII

## Some Theorems, associated with bilateral generating functions involving Hermite, Laguerre and Gegenbauer Polynomials.

1. INTRODUCTION : In the investigation of the generating relations group theoretic method seems to be a potent one in comparison with the analytic method because known generating functions can be verified and then extended by group theoretic method. This approach has been tried by S.K. Chatterjea [1] and many others.

In the present Chapter we shall adopt group theoretic method to obtain a new class of bilateral generating relations involving Generalized Hermite Polynomials. $H_n^r(x,a,p)$ [2], Laguerre Polynomial $L_n^{(\alpha)}(x)$ [3] and ultraspherical polynomial $C_n^{\lambda}(x)$ [3] in the present chapter we prove the following theorems :-

Theorem - 1 If there exists a bilateral generating relation of the form;-

$$G(x,z,\omega) = \sum_{K=0}^{\infty} \omega^K H_K^r(x,a,p) \cdot L_n^{(K)}(z)$$

then there exists a generating relation of the form.

$$x^{-a}(x+\omega)^a \exp\left[p\{x^r-(x+\omega)^r\}-\omega\right] \cdot G(x+\omega, z+\omega, \omega v)$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty} \omega^K f_K(\omega,v,x) \cdot L_n^{(K)}(z)$$

where

$$f_K(\omega,v,x) = \sum_{\mu=0}^{\min(K,\lambda)} \frac{(-1)^\lambda v^{K-\mu} \omega^{\lambda-\mu}}{\lfloor\mu \; \lfloor\lambda-\mu} H_{K+\lambda-2\mu}^r(x,a,p)$$

<u>Theorem-II</u> If there exists a bilateral generating relation of the form

$$G(x,z,w) = \sum_{\lambda=0}^{\infty} w^{\lambda} C_n^{\lambda}(x) \, L_n^{(\lambda)}(z)$$

then there exists a generating relation of the form.

$$\frac{\exp(-w)}{(\sqrt{1-zw})^{\alpha}} \, G\left(\frac{x}{\sqrt{1-zw}},\ z+w,\ \frac{wv}{1-zw}\right)$$

$$= \sum_{\lambda=0}^{\infty} \sum_{m=0}^{\infty} w^{\lambda} f_{\lambda}(w,v,x) \, . \, L_n^{(\lambda)}(z)$$

where

$$f_{\lambda}(w,v,x) = \sum_{p=0}^{\min(\lambda,m)} \frac{(-1)^p \, w^{m-p} \, v^{\lambda-p} \, z^{m-p} (\lambda)_{m-p}}{\underline{|m-p} \ \underline{|p}} \, . \, C_n^{\lambda+m-2p}(x)$$

## 7.2) <u>GROUP THEORETIC METHOD TO PROVE THEOREM I:-</u>

For the generalised Hermite Polynomials $H_K^r(x,a,p)$ defined by Gould & Hopper [2]

7.2.1) $$H_K^r(x,a,p) = (-1)^K x^{-a} e^{px^r} \, . \, D^K\left(x^a e^{-px^r}\right)$$

We consider the operator $\mathcal{C}_1$ where

7.2.2) $$\mathcal{C}_1 = y\,\mathfrak{D} = y\left[\frac{\partial}{\partial x} - prx^{r-1} + \frac{a}{x}\right]$$

Such that

7.2.3) $$\mathcal{C}_1\left[H_K^r(x,a,p)\, y^K\right] = -H_{K+1}^r(x,a,p) \, . \, y^{K+1}$$

The corresponding extended form of the group generated by $\mathcal{C}_1$ is found by solving following equations

(i) $$\frac{d}{dw} x(w) = y$$

$$d\,x(w) = y\int dw + K$$

$$x(w) = yw + K$$

when $w = 0$ $\quad x(0) = x$ $\quad$ then $k = x$

and $$x(w) = yw + x$$

(ii) $$\frac{d}{dw} v(w) = \left[-y\,pr\{x(w)\}^{r-1} + ay\{x(w)\}^{-1}\right] v(w)$$

$$\frac{d\,v(w)}{v(w)} = \int\left\{-ypr(x+wy)^{r-1} + ay(x+wy)^{-1}\right\} dw + K$$

$$\log v(w) = -ypr\frac{(x+wy)^r}{ry} + a\log(x+wy) + K$$

when $w = 0$, $\quad v(0) = 1$, then $K = px^r - a\log x$

$$\log v(w) = -p(x+wy)^r + a\log(x+wy) + px^r - a\log x$$

$$v(w) = x^{-a}(x+wy)^a e^{p\{x^r - (x+wy)^r\}}$$

Hence

7.2.4) $$\exp w\mathcal{C}_1\, f(x,y) = x^{-a}(x+wy)^a \cdot e^{p\{x^r - (x+wy)^r\}} \cdot f(x+wy, y)$$

for the Laguerre Polynomials $L_n^{(K)}(z)$ defined by [3]

7.2.5) $$L_n^{(K)}(z) = \sum_{S=0}^{n} \frac{(1+K)_n\,(-z)^S}{\lfloor S\ \lfloor n-S\ (1+K)_S}$$

We consider the operator $\mathcal{C}_2$ given by

7.2.6) $$\mathcal{C}_2 = t\frac{\partial}{\partial z} - t$$

Such that

7.2.7) $$\mathcal{E}_2\left[L_n^{(K)}(z).t^K\right] = -L_n^{(K+1)}(z).t^{K+1}$$

The corresponding extended form of the group generated by $\mathcal{E}_2$ is found by solving following equations.

(i) $$\frac{d}{dw}Z(\omega) = t$$

$$Z(\omega) = \int t.d\omega + K$$

$$Z(\omega) = t\omega + K$$

when $w = 0$ $\quad z(0) = z \quad$ then $K = z$

$$Z(\omega) = t\omega + Z$$

and

(ii) $$\frac{d}{dw}v(\omega) = -t\,v(\omega)$$

$$\int\frac{d\,v(\omega)}{v(\omega)} = -\int t.d\omega + K$$

$$\log v(\omega) = -t\omega + K$$

when $w = 0$, $\quad v(0) = 1$, $\quad K = 0$

$$v(\omega) = e^{-t\omega}$$

Hence

7.2.8) $$\exp\omega\mathcal{E}_2\ f(z,t) = e^{-t\omega}f(z+\omega t,t)$$

Consider the following bilateral generating relations

7.2.9) $$G(x,z,\omega) = \sum_{K=0}^{\infty}\omega^K H_K^{r}(x,a,p).L_n^{(K)}(z)$$

put $\omega = \omega y t v$ in (7.2.9) we get

7.2.10) $$G(x,z,\omega y t v) = \sum_{K=0}^{\infty}\left\{H_K^{r}(x,a,p)\ y^K\right\}.$$

$$.\left\{L_n^{(K)}(z)\,t^K\right\}.(\omega v)^K$$

Operating both sides of (7.2.10) by $(\exp\omega\mathcal{E}_1)(\exp\omega\mathcal{E}_2$

we get

$$(\exp w\mathcal{C}_1)(\exp w\mathcal{C}_2)\, G(x,z,wytv)$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty} \frac{(w\mathcal{C}_1)^{\lambda}}{\underline{|\lambda}} \cdot \frac{(w\mathcal{C}_2)^{\mu}}{\underline{|\mu}} \cdot$$

$$\left\{H_K^{r}(x,a,p)\, y^K\right\}\cdot\left\{L_n^{(K)}(z)\, t^K\right\}\cdot (wv)^K$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty} \frac{w^{\lambda+\mu+K}\, v^K}{\underline{|\lambda}\;\underline{|\mu}} (-1)^{\lambda+\mu} y^{K+\lambda}$$

$$\cdot H_{K+\lambda}^{r}(x,a,p)\; L_n^{K+\mu}(z)\; t^{K+\mu}$$

But

$$(\exp w\mathcal{C}_1)(\exp w\mathcal{C}_2)\, G(x,z,wytv)$$

$$= x^{-a}(x+wy)^a \exp\left[p\{x^r-(x+wy)^r\}\right]\cdot$$

$$\cdot \exp(-wt)\, G(x+wy, z+wt, wytv)$$

Hence from these two expressions

7.2.11)
$$x^{-a}(x+wy)^a \exp\left[p\{x^r-(x+wy)^r\}-wt\right]\cdot$$

$$\cdot\, G(x+wy,\ z+wt,\ wytv)$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\min(K,\lambda)} \frac{v^{K-\mu}\, w^{\lambda+K-\mu}\,(-1)^{\lambda}\, t^K}{\underline{|\mu}\;\underline{|\lambda-\mu}}\cdot$$

$$\cdot\, y^{K+\lambda-2\mu}\; H_{K+\lambda-2\mu}^{r}(x,a,p)\; L_n^{(K)}(z)$$

By putting $t=y=1$ on both sides we get

$$x^{-a}(x+w)^a \exp\left[p\{x^r-(x+w)^r\}-w\right]\cdot$$

$$G(x+w,\ z+w,\ wv)$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\min(K,\lambda)} \frac{v^{K-\mu}\, w^{\lambda+K-\mu}\,(-1)^{\lambda}}{\underline{|\mu}\;\underline{|\lambda-\mu}}\cdot$$

$$\cdot\, H_{K+\lambda-2\mu}^{r}(x,a,p)\cdot L_n^{(K)}(z)$$

$$= \sum_{K=0}^{\infty}\sum_{\lambda=0}^{\infty} w^K\, f_K(w,v,x)\; L_n^{(K)}(z)$$

where

$$f_K(\omega, v, x) = \sum_{\mu=0}^{\min(K,\lambda)} \frac{(-1)^{\lambda} v^{K-\mu} \omega^{\lambda-\mu}}{\underline{|\mu} \; \underline{|\lambda-\mu}} \cdot H^{\gamma}_{K+\lambda-2\mu}(x, a, p)$$

this proves the Theorem - I

7.3) <u>Proof of Theorem -II</u> :-

The Gegenbauer polynomials definied by [3]

7.3.1) $$C_n^{\lambda}(x) = \sum_{K=0}^{[n/2]} \frac{(-1)^K (\lambda)_{n-K} (2x)^{n-K}}{\underline{|K} \; \underline{|n-2K}}$$

Consider the operator $\mathcal{C}_1$ where

7.3.2) $$\mathcal{C}_1 = xy\frac{\partial}{\partial x} + 2y^2\frac{\partial}{\partial y} + \alpha y$$

Such that

7.3.3) $$\mathcal{C}_1\left\{C_n^{\lambda}(x) . y^{\lambda}\right\} = 2\lambda\, C_n^{\lambda+1}(x)\, y^{\lambda+1}$$

The corresponding extended form of the group generated by $\mathcal{C}_1$ is found by solving following equations.

(i) $$\frac{d}{d\omega} x(\omega) = x(\omega)\, y(\omega)$$

(ii) $$\frac{d}{d\omega} y(\omega) = 2\{y(\omega)\}^2$$

(iii) $$\frac{d}{d\omega} v(\omega) = \alpha\, y(\omega) . v(\omega)$$

By solving (ii) equation we get

$$\int \frac{d\,y(\omega)}{\{y(\omega)\}^2} = 2\int d\omega + K$$

$$-\frac{1}{y(\omega)} = 2\omega + K$$

when w = 0, $y(0) = y$ , then $K = -\frac{1}{y}$

$$-\frac{1}{y(\omega)} = 2\omega - \frac{1}{y}$$

$$y(\omega) = \frac{y}{1-2\omega y}$$

By putting the value of y(w) in (i) we get

$$\frac{d}{d\omega} x(\omega) = x(\omega) \cdot \frac{y}{1-2\omega y}$$

$$\log x(\omega) = -\frac{1}{2}\log(1-2\omega y) + K$$

when $w = 0$, $x(o) = x$ then $K = \log x$

$$x(\omega) = \frac{x}{\sqrt{1-2\omega y}}$$

By solving (iii) we get

$$\int \frac{dv(\omega)}{v(\omega)} = \int \frac{\alpha y}{1-2\omega y}\, d\omega + K$$

$$\log v(\omega) = -\frac{\alpha}{2}\log(1-2\omega y) + K$$

when $w = 0$, $v(o) = 1$ then $K = 0$

$$v(\omega) = \frac{1}{(\sqrt{1-2\omega y})^{\alpha}}$$

Hence

7.3.4) $$(\exp \omega e_1)\, f(x,y) = \frac{1}{(\sqrt{1-2\omega y})^{\alpha}}\, f\left\{\frac{x}{\sqrt{1-2\omega y}}, \frac{y}{1-2\omega y}\right\}$$

Using equations (7.2.5) (7.2.6) (7.2.7) (7.2.8) we consider the following bilateral generating relation

7.3.5) $$G(x,z,\omega) = \sum_{\lambda=0}^{\infty} \omega^{\lambda}\, C_n^{\lambda}(x)\, L_n^{(\lambda)}(z)$$

Putting $w = wytv$ we get

7.3.6) $$G(x,z,\omega y t v) = \sum_{\lambda=0}^{\infty} \left(C_n^{\lambda}(x)\, y^{\lambda}\right)\left(L_n^{(\lambda)}(z)\, t^{\lambda}\right) \cdot (\omega v)^{\lambda}$$

Operating both sides by $(\exp \omega e_1)(\exp \omega e_2)$ and using the method as in section (2), we get

$$\frac{\exp(-\omega)}{(\sqrt{1-2\omega})^{\alpha}}\, G\left(\frac{x}{\sqrt{1-2\omega}},\ z+\omega,\ \frac{\omega v}{1-2\omega}\right)$$

$$= \sum_{\lambda=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\min(\lambda,m)} \frac{(-1)^{p}\,\omega^{\lambda+m-p}\,v^{\lambda-p}}{\lfloor m-p\ \lfloor p} \cdot$$

$$\cdot\, z^{m-p}\,(\lambda)_{m-p}\, C_n^{\lambda+m-2p}(x)\ L_n^{(\lambda)}(z)$$

$$= \sum_{\lambda=0}^{\infty}\sum_{m=0}^{\infty} \omega^{\lambda} \cdot f_{\lambda}(\omega, v, x) \cdot L_n^{\lambda}(z)$$

where

$$f_{\lambda}(\omega, v, x) = \sum_{p=0}^{\min(\lambda,m)} \frac{(-1)^{p}\,\omega^{m-p}\,v^{\lambda-p}}{\lfloor m-p\ \lfloor p}\, z^{m-p} \cdot (\lambda)_{m-p}\, C_n^{\lambda+m-2p}(x)$$

This proves the Theorem II

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# CHAPTER VIII

## LIE OPERATORS & GENERALISED BESSEL POLYNOMIALS.

1. INTRODUCTION - Krall and Frink [1] initiated the study of the Bessel Polynomials. In their terminology the simple Bessel polynomial is

8.1.1) $$y_n(x) = {}_2F_0\left[-n, 1+n; -; -\tfrac{1}{2}x\right]$$

and the generalised one is

8.1.2) $$y_n(x, a, b) = {}_2F_0\left[-n, a-1+n; -; -\frac{x}{b}\right]$$

in the present chapter the authors will apply the Lie - Group theory to obtain a class of generating functions for $y_n(x, a, b)$ and other relations.

8.2) Differential Equation for $y_n(x, a, b)$ :- R.P. Agarwal [2] has given following two differential recurrence relations for $y_n(x, a, b)$

8.2.1) $$\left[x^2(2n+a-2)D - n(2n+a-2)x - nb\right]y_n(x,a,b) = nb\, y_{n-1}(x,a,b)$$

8.2.2) $$\left[(2n+a)x^2 D - (1-n-a)\{(2n+a)x+b\}\right] \cdot y_n(x,a,b) = b(n+a-1)\, y_{n+1}(x,a,b)$$

From (8.2.1) and (8.2.2) we get the following differential equation for

8.2.3) $$\left[(2n-2+a)x^2 D - (2-n-a)\{(2n-2+a)x+b\}\right] \cdot \left[(2n+a-2)x^2 D - n(2n+a-2)x - nb\right] y_n(x,a,b) - nb^2(n+a-2)\, y_n(x,a,b) = 0$$

Replacing n by $y\frac{\partial}{\partial y}$ and D by $\frac{\partial}{\partial x}$ we get the partial differential equation satisfied by

$$u(x,y) = y^n \cdot y_n(x,a,b)$$

8.2.4) $$L\,u(x,y) = 0$$

Now consider the following differential operators :

8.2.5) $$A = y\frac{\partial}{\partial y}$$

$$B = 2x^2\frac{\partial^2}{\partial x\,\partial y} - 2yx\frac{\partial^2}{\partial y^2} + (a-2)\frac{x^2}{y}\frac{\partial}{\partial x} - (ax+b)\frac{\partial}{\partial y}$$

$$\mathcal{C} = 2y^2x^2\frac{\partial^2}{\partial x\,\partial y} + 2xy^3\frac{\partial^2}{\partial y^2} + ax^2y\frac{\partial}{\partial x} + (3ax+b)y^2\frac{\partial}{\partial y} + (ax+b)(a-1)y$$

Then

8.2.6) $$L \equiv \mathcal{C}B - b^2A(A+a-2)$$

8.2.7) $$B\{y^n \cdot y_n(x,a,b)\} = nb\,y^{n-1} \cdot y_{n-1}(x,a,b)$$

$$\mathcal{C}\{y^n \cdot y_n(x,a,b)\} = b(n+a-1)\,y^{n+1} \cdot y_{n+1}(x,a,b)$$

8.2.8) $$[A,B] = -B \quad , \quad [A,\mathcal{C}] = \mathcal{C}$$

$$[B,\mathcal{C}] = (2A+a-1)\,b^2$$

where $[A,B]u = (AB - BA)u$ etc

These commutator relations show that

1, A, B, $\mathcal{C}$, generate a Lie group.

By putting

8.2.9) $$B_1 = (2n+a-2)\frac{x^2}{y}\frac{\partial}{\partial x} - n(2n+a-2)\frac{x}{y} - \frac{nb}{y}$$

$$\mathcal{C}_1 = y(2n+a)x^2\frac{\partial}{\partial x} - y(1-n-a)\{(2n+a)x+b\}$$

and by using the standard lie theoretic techniques by Miller (3) we express the extended forms of the group generated by A, as follows :

8.2.10) $\exp pA \; f(x,y) = f(x, ye^{p})$

To find extended forms of the group generated by B we have to solve following equations.

(i) $$\frac{d}{dw} x(w) = \frac{(2n+a-2)}{y} x^2(w)$$

$$\int \frac{dx(w)}{x^2(w)} = \int \frac{(2n+a-2)}{y} dw + K$$

$$-\frac{1}{x(w)} = \frac{(2n+a-2)}{y} w + K$$

When w = 0 $\quad x(0) = 0$ $\quad$ then $K = -\frac{1}{x}$

$$-\frac{1}{x(w)} = \frac{2n+a-2}{y} w - \frac{1}{x}$$

$$x(w) = \frac{x}{1-(2n+a-2)\frac{wx}{y}}$$

and

(ii) $$\frac{dv(w)}{dw} = \left\{-\frac{nb}{y} - \frac{n(2n+a-2)x(w)}{y}\right\} v(w)$$

$$\int \frac{dv(w)}{v(w)} = \int\left[-\frac{nb}{y} - \frac{n(2n+a-2)x}{\{y-(2n+a-2)wx\}}\right] dw$$

$$\log v(w) = -\frac{nbw}{y} + n\log\left[y-(2n+a-2)wx\right] + K$$

When $w = 0$ , $v(0) = 1$ , then $K = -n\log y$

$$\log v(w) = -\frac{nbw}{y} + n\log\left\{1-(2n+a-2)\frac{wx}{y}\right\}$$

$$v(w) = e^{-\frac{nbw}{y}}\left\{1-(2n+a-2)\frac{wx}{y}\right\}^{n}.$$

Hence

8.2.11) $\exp wB\left[y^n f(x)\right] = y^n e^{-\frac{nbw}{y}}.$

$$\left\{1-(2n+a-2)\frac{\omega x}{y}\right\}^{n}.$$
$$\cdot f\left\{\frac{x}{1-(2n+a-2)\frac{\omega x}{y}}\right\}$$

To find extended forms of the group generated by $\mathcal{C}$ we have to solve following equations.

$$\frac{dx(c)}{dc} = y(2n+a)x^2(c)$$

$$\int \frac{dx(c)}{x^2(c)} = \int y(2n+a)\,dc + K$$

$$-\frac{1}{x(c)} = (2n+a)yc + K$$

When $c = 0$, $x(0) = x$ then $K = -\frac{1}{x}$

$$x(c) = \frac{x}{1-(2n+a)cxy}$$

and

(ii) $$\frac{dv(c)}{dc} = -y(1-n-a)\left\{(2n+a)x(c)+b\right\}v(c)$$

$$\int \frac{dv(c)}{v(c)} = -\int y(1-n-a)\left\{(2n+a)\frac{x}{1-(2n+a)cxy} + b\right\}dc + K.$$

$$\log v(c) = (1-n-a)\log\left\{1-(2n+a)cxy\right\}.$$
$$-byc(1-n-a)$$

Hence $$v(c) = e^{-byc(1-n-a)}\cdot\left\{1-(2n+a)cxy\right\}^{1-n-a}.$$

8.2.12) $$\exp c\mathcal{C}\{y^n f(x)\} = y^n e^{-byc(1-n-a)}.$$
$$\cdot\left\{1-(2n+a)cxy\right\}^{1-n-a}.$$
$$\cdot f\left\{\frac{x}{1-(2n+a)cxy}\right\}$$

8.3) <u>Generating Functions Anulled by conjugates of (A - n):-</u> we see that $y^n.y_n(x,a,b)$ are solutions of the differential equations $L(u) = 0$ and $Au = nu$ for arbitrary n. Now

8.3.1) $$e^{\omega B}e^{c\mathcal{C}}\left[y^n.y_n(x,a,b)\right]$$

$$= e^{(n+a-1)bcy - \frac{nbw}{y}} \left\{1-(2n+a)cxy\right\}^{1-2n-a} \cdot$$

$$\cdot \left\{y-(2n+a)cxy^2-(2n+a-2)wx\right\}^n$$

$$\cdot \left[y_n\left\{\frac{xy}{y-(2n+a)cxy^2-(2n+a-2)wx}\right\}, a, b\right]$$

$$= G(x,y)$$

Put $S = e^{bB+cG}$ then $SAS^{-1}$ is conjugate of $A$ and $G(x,y)$ is annulled by $L$ and $S(A-n)S^{-1}$

we consider the following cases:

Case - I : $c = 0$ $w = 1$

then (8.3.1) reduces to

8.3.2)
$$e^{B}\left[y^n . y_n(x,a,b)\right]$$
$$= y^n\left\{1-(2n+a-2)\frac{x}{y}\right\}^n e^{-\frac{nb}{y}} \cdot$$
$$\cdot y_n\left\{\frac{x}{1-(2n+a-2)\frac{x}{y}}, a, b\right\}$$

8.3.3) Also
$$e^{B}\left[y^n . y_n(x,a,b)\right] = \sum_{K=0}^{\infty} \frac{B^K}{\lfloor K}\left\{y^n . y_n(x,a,b)\right\}$$
$$= \sum_{K=0}^{\infty} \frac{B^{K-1}}{\lfloor K} nb\, y^{n-1} . y_{n-1}(x,a,b)$$

$$= \sum_{K=0}^{\infty} \frac{B^{K-K}}{\lfloor K}\left\{nb.(n-1)b --- (n-K+1)b\right\}.$$
$$\cdot y^{n-K} y_{n-K}(x,a;b)$$
$$= \sum_{K=0}^{\infty} \frac{b^K}{\lfloor K} \frac{\lfloor n}{\lfloor n-K} y^{n-K} y_{n-K}(x,a,b)$$
$$= y^n \sum_{K=0}^{n} \binom{n}{K}\left(\frac{b}{y}\right)^K y_{n-K}(x,a,b)$$

as $y_{-p} = 0$

Equating the two values and after minor adjustments we get

8.3.4) $$(1-tx)^n \, e^{-\frac{nbt}{2n+a-2}} \, y_n\left\{\frac{x}{1-xt}, a, b\right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{(bt)^k}{(2n+a-2)^k} \, y_{n-k}(x, a, b)$$

Where $t = \dfrac{2n+a-2}{y}$

<u>Special Case</u> :- Put a = b = 2 in (8.3.4) we get

8.3.5) $$(1-tx)^n \, e^{-t} \, y_n\left\{\frac{x}{1-xt}\right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{n^k} \cdot y_{n-k}(x)$$

<u>Case II</u> :- w = 0, c = 1

Then ( 8.3.1) reduces to

8.3.6) $$e^{\mathcal{E}}\left\{y^n . y_n(x, a, b)\right\} = y^n \, e^{(n+a-1)by} \cdot \left\{1-(2n+a)xy\right\}^{1-n-a} \cdot y_n\left\{\frac{x}{1-(2n+a)xy}, a, b\right\}$$

But

8.3.7) $$e^{\mathcal{E}}\left\{y^n . y_n(x, a, b)\right\} = \sum_{k=0}^{\infty} \frac{(\mathcal{E})^k}{k!}\left[y^n . y_n(x,a,b)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(\mathcal{E})^{k-1}}{k!} \, b(n+a-1) \, y^{n+1} . y_{n+1}(x,a,b)$$

$$- - -$$

$$= \sum_{k=0}^{\infty} \frac{(\mathcal{E})^{k-k}}{k!} \left\{b(n+a-1) . b(n+a) \cdots b(n+a+k-2)\right\} y^{n+k} . y_{n+k}(x,a,b)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \, b^k \, \frac{(n+a+k-2)!}{(n+a-2)!} \, y^{n+k} \, y_{n+k}(x,a,b)$$

$$= y^n \sum_{k=0}^{\infty} \binom{n+k+a-2}{k} (by)^k \, y_{n+k}(x,a,b)$$

Equating both values and after minor adjustments we get

$$8.3.8)\quad \{1-tx\}^{1-n-a} \cdot e^{(n-a-1)\frac{bt}{2n+a}} \cdot y_n\left\{\frac{x}{1-tx}, a, b\right\}$$

$$= \sum_{K=0}^{\infty} \binom{n+K+a-2}{K} \frac{(bt)^K}{(2n+a)^K}\, y_{n+K}(x, a, b)$$

Where $t = (2n+a)y$

Special Case :- By putting a =b = 2 in (8.3.8) we get

$$8.3.9)\quad (1-tx)^{-n-1}\, e^{t} \cdot y_n\left\{\frac{x}{1-tx}\right\}$$

$$= \sum_{K=0}^{\infty} \binom{n+K}{K} \frac{t^K}{(n+1)^K}\, y_{n+K}(x)$$

Case III :- $w \neq 0$, $c = 1$

then (8.3.1 ) reduces to

$$8.3.10)\quad e^{\omega B} e^{G}\left[y^n \cdot y_n(x,a,b)\right]$$

$$= e^{(n+a-1)by - \frac{nb\omega}{y}} \cdot \{1-(2n+a)xy\}^{1-2n-a} \cdot$$

$$\cdot \{y-(2n+a)xy^2 - (2n+a-2)\omega x\}^n$$

$$\cdot y_n\left\{\frac{xy}{y-(2n+a)xy^2-(2n+a-2)\omega x}, a, b\right\}$$

Also

$$8.3.11)\quad e^{\omega B} e^{G}\left[y^n \cdot y_n(x,a,b)\right]$$

$$= e^{\omega B}\, y^n \sum_{K=0}^{\infty} \binom{n+K+a-2}{K} (by)^K y_{n+K}(x,a,b)$$

$$= \sum_{K=0}^{\infty} \sum_{S=0}^{n+K} \binom{n+K+a-2}{K} b^K \binom{n+K}{S} (\omega b)^S \cdot$$

$$\cdot y^{n+K-S}\, y_{n+K-S}(x,a,b)$$

Equating both values and after minor adjustments we get

8.3.12)
$$e^{(n+a-1)by - \frac{nbw}{y}} \cdot \left\{1-(2n+a)xy\right\}^{1-2n-a}$$
$$\cdot \left\{1-(2n+a)xy-(2n+a-2)\frac{wx}{y}\right\}^{n}$$
$$\cdot y_n\left\{\frac{x}{1-(2n+a)xy-(2n+a-2)\frac{wx}{y}}, a, b\right\}$$
$$= \sum_{K=0}^{\infty}\sum_{S=0}^{n+K}\binom{n+K+a-2}{K} b^K \binom{n+K}{S}(wb)^S y^{K-S} \cdot$$
$$\cdot y_{n+K-S}(x,a,b)$$

8.3.13) for a=b=2 (8.3.12) reduces to
$$e^{2(n+1)y-\frac{2nw}{y}} \left\{1-2(n+1)xy\right\}^{-1-2n} \cdot$$
$$\cdot \left\{1-2(n+1)xy-\frac{2nwx}{y}\right\}^{n}$$
$$\cdot y_n\left\{\frac{x}{1-2(n+1)xy-\frac{2nwx}{y}}\right\}$$
$$= \sum_{K=0}^{\infty}\sum_{S=0}^{n+K}\binom{n+K}{K}\binom{n+K}{S} w^S 2^{K+S} y^{K-S} \cdot$$
$$\cdot y_{n+K-S}(x)$$

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# CHAPTER IX

## LIE OPERATORS AND GENERALIZED HERMITE FUNCTIONS

9.1) INTRODUCTION : Usually the Hermite polynomials are defined by the relation -

9.1.1) $$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} \qquad D \equiv \frac{\partial}{\partial x}$$

The second way of defining the Hermite polynomials, which is not very common is

9.1.2) $$H_n\left(\frac{x}{2}\right) = e^{-D^2} \cdot x^n$$

Gould-Hopper [2] generalised (9.1.1) and (9.1.2) both and gave explicit forms for the generalised functions by the following relations respectively.

9.1.3) $$H_n^r(x,a,p) = (-1)^n x^{-a} e^{px^r} D^n \left(x^a e^{-px^r}\right)$$

$$= (-1)^n \lfloor n \sum_{K=0}^{n} \frac{p^K x^{rK-n}}{\lfloor K} \sum_{j=0}^{K} (-1)^j \binom{K}{j} \binom{a+rj}{n}$$

and

9.1.4) $$g_n^r(x,h) = e^{hD^r} x^n.$$

$$= \sum_{K=0}^{[n/r]} \frac{\lfloor n \, h^K}{\lfloor K \lfloor n-rK} x^{n-rK}$$

The respective generating relations are

9.1.5) $$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n^r(x,a,p) = x^{-a} (x-t)^a e^{p\{x^r - (x-t)^r\}}$$

and

9.1.6) $$\sum_{n=0}^{\infty} g_n^r(x,h) \frac{t^n}{\lfloor n} = e^{tx + ht^r}$$

In the present note the authors have applied the Lie-Group theory to obtain certain generating relations. for $g_n^r(x,h)$

9.2) DIFFERENTIAL EQUATION AND LIE OPERATORS :

From [2] we see that $g_n^r(x,h)$ satisfies the following differential equation

9.2.1) $h r D^r g_n^r + x D g_n^r - n g_n^r = 0 \quad , r \geq 1$

Replacing D by $\frac{\partial}{\partial x}$ and n by $y\frac{\partial}{\partial y}$ we get the rth order partial differential equation satisfied by $u(x,y) = y^n . g_n^r(x,h)$ as

9.2.2) $$L u(x,y) = \left[ h r \frac{\partial^r}{\partial x^r} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right] u(x,y) = 0$$

consider the following infinitesimal operators

9.2.3) $$A = y\frac{\partial}{\partial y} , \quad B = \frac{1}{y}\frac{\partial}{\partial x} , \quad C = yx + y h r \frac{\partial^{r-1}}{\partial x^{r-1}}$$

Then we see that

9.2.4) $L \equiv CB - A$ and the commutator relations satisfied by A, B, & C are

9.2.5) $$[A,B] = -B \quad , \quad [A,C] = C \quad ,$$
$$[B,C] = 1$$

Thus we find that 1, A, B and C generate a Lie group $\Gamma$

To find extended form of the transformation group generated by A we have to solve following equations

$$\frac{\partial}{\partial a} y(a) = y(a)$$

$$\int \frac{\partial y(a)}{y(a)} = \int \partial a + K$$

$$\log y(a) = a + K$$

When a=0, y(o)=y, then K = log y

$$y(a) = y e^a$$

Hence

9.2.6) $$e^{aA} f(x,y) = f(x, y e^a)$$

To find extended form of the transformation group generated by $B$ we have to solve following equations.

$$\frac{\partial}{\partial b} x(b) = \frac{1}{y}$$

$$\int \partial x(b) = \int \frac{1}{y} \partial b + K$$

$$x(b) = \frac{b}{y} + K$$

when $b = 0$, $x(0) = x$ then $K = x$

$$x(b) = x + \frac{b}{y}$$

Hence

9.2.7) $$e^{bB} f(x,y) = f\left(x + \frac{b}{y}, y\right)$$

To obtain $e^{cG} f(x,y)$ we use the following operational relations [2]

9.2.8) $$\left[x + hr D_x^{r-1}\right]^n f(x) = e^{hD_x^r} . x^n \left(e^{-hD_x^r} f(x)\right)$$

Now

$$e^{cG} f(x,y) = \sum_{n=0}^{\infty} \frac{c^n G^n}{\lfloor n} f(x,y)$$

where $G = y(x + hr D^{r-1})$

so the expression becomes

$$= \sum_{n=0}^{\infty} \frac{c^n y^n (x + hr D^{r-1})^n}{\lfloor n} f(x,y)$$

$$= \sum_{n=0}^{\infty} e^{hD_x^r} \left(\frac{c^n y^n x^n}{\lfloor n}\right) e^{-hD_x^r} f(x,y)$$

9.2.9) $$e^{cG} f(x,y) = e^{hD_x^r} e^{cxy} e^{-hD_x^r} f(x,y)$$

$$D_x \equiv \frac{\partial}{\partial x}$$

Now from (9.2.7) and (9.2.9) we get

9.2.10) $$e^{bB + cG} f(x,y)$$

$$= e^{hD_z^r} \left[e^{cyz} \left(e^{-hD_z^r} f(z,y)\right)\right]$$

Where $z = x + \frac{b}{y}$

In particular we note that

9.2.11) $A\left[y^n . g_n^r(x,h)\right] = n y^n . g_n^r(x,h)$

9.2.12) $B\left[y^n . g_n^r(x,h)\right] = n y^{n-1} . g_{n-1}^r(x,h)$

9.2.13) $C\left[y^n . g_n^r(x,h)\right] = y^{n+1} . g_{n+1}^r(x,h)$

9.3) Conjugate Operators : We shall examine the functions annulled by L and $R = r_1 A + r_2 B + r_3 C + r_4$ where r's are arbitrary constants and $r_1, r_2, r_3$ do not vanish simultaneously. Now we shall separate the operator R into conjugate classes with respect to the group. $\Gamma^{+}$

Using the method of Weisner [4] we have -

$$e^{bB} A e^{-bB} = \sum_{K=0}^{\infty} \frac{b^K}{\underline{|K}} [B, A]_K$$

$$= [B, A]_0 + b [B, A]_1 + \frac{b^2}{\underline{|2}} [B, A]_2 + \text{---}$$

Now

$$[B, A]_0 = A$$

$$[B, A]_1 = BA - AB = B$$

$$[B, A]_2 = [B, [B, A]_1]$$

$$= [B, B] = 0$$

Hence the expression becomes

$$e^{bB} A e^{-bB} = A + bB$$

similarly we can find other expressions

9.3.1) $e^{aA} B e^{-aA} = e^{-a} B$

$$e^{bB} A e^{-bB} = A + bB$$

$$e^{cC} B e^{-cC} = B + c$$

$$e^{aA} C e^{-aA} = e^{a} C$$

$$e^{bB} C e^{-bB} = C + b$$

$$e^{c\varphi} A e^{-c\varphi} = A - c\varphi$$

Now setting $S = e^{bB + c\varphi}$, we have

$$SAS^{-1} = A + bB - c\varphi - bc$$

$$SBS^{-1} = B + c$$

9.3.2) $$S\varphi S^{-1} = \varphi + b.$$

From these formulae it follows that R is conjugate to

(a) $A - n$ if $r_1 = 1$

(b) $B + c$ if $r_1 = 0$, $r_3 = 0$, $r_2 \neq 0$

(c) $\varphi + b$ if $r_1 = 0$, $r_2 = 0$, $r_3 \neq 0$

The identity (9.2.4) shows that the cases (b) and (c) do not require special attention.

## 9.4) GENERATING FUNCTIONS FOR FUNCTIONS ANNULLED BY CONJUGATES OF $(A - n)$ :-

Since $u = y^n . g_n^r(x,h)$ is a solution of Simultaneous equations $Lu = 0$ and $(A - n)u = 0$ where n is an arbitrary constant, it follows from (9.2.10) that

$$G(x,y) = y^n e^{hD_z^r}\left[e^{cyz}\left(e^{-hD_z^r} . g_n^r(z,h)\right)\right]$$

where $z = x + \frac{b}{y}$

is a solution of $Lu = 0$ and $\{S(A-n)S^{-1}\}u = 0$ so we examine G.

<u>Case I</u> : $C = 0$ : setting $b = 1$ and $t = 1/y$, we obtain after simplification

9,4.1) $$g_n^r(x+t, h) = \sum_{i=0}^{\infty} \binom{n}{i} t^i g_{n-i}^r(x,h)$$

a Taylor's expansion which may be derived from [2]

$$D_x\, g_n^r(x,h) = n\, g_{n-1}^r(x,h)$$

<u>Case 2</u> : b = 0 : Setting c = 1 , we have

$$e^{\mathcal{G}}\left[y^n . g_n^r(x,h)\right] = \sum_{j=0}^{\infty} \frac{(\mathcal{G})^j}{\lfloor j}\left[y^n . g_n^r(x,h)\right]$$

$$= \sum_{j=0}^{\infty} \frac{(\mathcal{G})^{j-1}}{\lfloor j}\left[y^{n+1} . g_{n+1}^r(x,h)\right]$$

using (9.2.13)

$$- . - - -$$

$$= \sum_{j=0}^{\infty} \frac{(\mathcal{G})^{j-j}}{\lfloor j}\left[y^{n+j}\, g_{n+j}^r(x,h)\right]$$

$$= \sum_{j=0}^{\infty} g_{n+j}^r(x,h) . \frac{y^{n+j}}{\lfloor j}$$

From (9.2.10) we get

$$e^{\mathcal{G}}\left[y^n . g_n^r(x,h)\right] = y^n e^{hD_x^r} . \left[e^{yx}\left(e^{-hD_x^r} g_n^r(x,h)\right)\right]$$

where $f(x,y) = y^n\, g_n^r(x,h)$

on comparing both values of $e^{\mathcal{G}}\left[y^n . g_n^r(x,h)\right]$

we get

$$\text{9.4.2)}\quad \sum_{j=0}^{\infty} g_{n+j}^r(x,h) . \frac{y^{n+j}}{\lfloor j} = y^n e^{hD_x^r} . \left[e^{yx}\left(e^{-hD_x^r} . g_n^r(x,h)\right)\right]$$

$$= y^n e^{hD_x^r}\left[x^n e^{yx}\right]$$

Using the inverse relations in [2]

$$\text{9.4.3)}\quad e^{-hD_x^r}\, g_n^r(x,h) = x^n$$

R.H.S of (9.4.2) further simplfying to

$$= y^n e^{xy + hy^r} \sum_{j=0}^{n} \binom{n}{j} x^{n-j}\, H_j^r(y, 0, -h)$$

with the help of operational relations in [2]

$$e^{hD_x^r}\left[x^n e^{yx}\right] = D_y^n\left[e^{yx+hy^r}\right]$$
$$= e^{yx+hy^r}\left[x + hry^{r-1} + D_y\right]^n . 1$$
$$= e^{yx+hy^r}\sum_{j=0}^{n}\binom{n}{j} x^{n-j} .$$
$$\cdot (D_y + hry^{r-1})^j . 1$$
$$= e^{yx+hy^r}\sum_{j=0}^{n}\binom{n}{j} x^{n-j} H_j^r(y,0,-h)$$

Cancelling $y^n$ in (9.4.2) we get

$$\sum_{j=0}^{\infty}\frac{y^j}{\underline{|j}}\, g_{n+j}^r(x,h) = e^{xy+hy^r}\sum_{j=0}^{n}\binom{n}{j} x^{n-j} \cdot H_j^r(y,0,-h) \tag{9.4.4}$$

Using (.9.1.6), (9.4.4) can also be written as

$$\sum_{j=0}^{\infty} g_{n+j}^r(x,h) = \sum_{m=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} H_j^r(y,0,-h) \cdot y^m g_m^r(x,h) . x^{n-j} \tag{9.4.5}$$

<u>Case III</u> : b, c $\neq$ 0, Setting b = w and c = 1

$$e^{\omega\mathcal{B}+\mathcal{C}}\left\{y^n . g_n^r(x,h)\right\}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\omega^i y^{n+j-i}}{\underline{|i}\,\underline{|j}}\, g_{n+j-i}^r(x,h)$$

using (9.2.12) and (9.2.13)

on the other side

$$e^{\omega\mathcal{B}+\mathcal{C}}\left\{y^n . g_n^r(x,h)\right\}$$
$$= y^n e^{hD_z^r}\left[e^{yz}\left(e^{-hD_z^r} g_n^r(z,h)\right)\right] \quad \text{where } z = x + \frac{\omega}{y}$$
$$= y^n e^{hD_z^r}\left[e^{yz} . z^n\right] \quad \text{using (9.4.3)}$$
$$= y^n \sum_{l=0}^{\infty}\frac{y^l}{\underline{|l}}\left(e^{hD_z^r} z^{n+l}\right)$$
$$= y^n \sum_{l=0}^{\infty}\frac{y^l}{\underline{|l}}\, g_{n+l}^r\left(x + \frac{\omega}{y}, h\right)$$

Using the operational relations in [2]

$$g_n^r(x,h) = e^{hD^r} x^n$$

Equating both values of $e^{\omega B + G}\{y^n . g_n^r(x,h)\}$ and cancelling $y^n$, we get

9.4.6) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega^i y^{j-i}}{\lfloor i \lfloor j} g^r_{n+j-i}(x,h)$$

$$= \sum_{l=0}^{\infty} \frac{y^l}{\lfloor l} g^r_{n+l}\left(x+\frac{\omega}{y}, h\right)$$

9.5) <u>Special Cases</u> : From (9.1.2) we have

$$H_n(x/2) = g_n^2(x,-1)$$

Hence relations (9.4.1), (9.4.4) and (9.4.6) reduce to the generating relations for simple Hermite polynomials $H_n(x)$ as

9.5.1) $$\sum_{i=0}^{n} \binom{n}{i} (2y)^i H_{n-i}(x/2) = H_n(x/2 + y)$$

9.5.2) $$\sum_{j=0}^{\infty} \frac{y^j}{\lfloor j} H_{n+j}(x/2) = e^{xy-y^2} . \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{n-j} H_j(y)$$

$$= e^{xy-y^2} . H_n\left(\frac{x}{2} - y\right) \quad \text{by (9.5.1)}$$

9.5.3) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega^i y^{j-i}}{\lfloor i \lfloor j} H_{n+j-i}\left(\frac{x}{2}\right)$$

$$= \sum_{l=0}^{\infty} \frac{y^l}{\lfloor l} H_{n+l}\left\{\frac{1}{2}\left(x+\frac{\omega}{y}\right)\right\}$$

$$= e^{zy-y^2} H_n\left(\frac{z}{2} - y\right)$$

Where $z = x + \dfrac{\omega}{y}$

by (9.5.2)

(9.5.1) and (9.5.2) of the above are known relations.

9.6) We have

$$e^{txy} = \sum_{n=0}^{\infty} \frac{t^n x^n y^n}{\lfloor n}$$

$$e^{hD_x^r}\, e^{txy} = \sum_{n=0}^{\infty} \frac{t^n y^n}{\lfloor n} \{e^{hD_x^r}\, x^n\}$$

$$= \sum_{n=0}^{\infty} \frac{t^n y^n}{\lfloor n}\, g_n^r(x,h)$$

(9.6.1) $$= e^{tyx + ht^r y^r}$$

using (9.1.6)

$$e^{KD_y^s}\, e^{hD_x^r}\, e^{txy} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n}\, g_n^s(y,K)\, g_n^r(x,h)$$

$$= e^{KD_y^s}\left(e^{tyx+ht^r y^r}\right)$$

using (9.6.1)

$$= e^{KD_y^s}\left(f(y).\, e^{txy}\right)$$

where $f(y) = e^{ht^r y^r}$

$$= f(D_z)\left(e^{Kt^s x^s}\, e^{txy}\right)$$

where. $z = tx$

$$= \exp(ht^r.D_z^r)\left[e^{txy + Kt^s x^s}\right]$$

$$= \left[\exp(ht^r.D_z^r)\left(e^{yz + Kz^s}\right)\right]_{z=tx}$$

on using the following relation in [2]

$$e^{hD^r}\{f(x).\, e^{tx}\} = f(D_t)\{e^{ht^r + tx}\}$$

Thus we get extension of Mehler's formula as

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n}\, g_n^s(y,K)\, g_n^r(x,h)$$

$$= \left[e^{ht^r D_z^r}.\, e^{yz + Kz^s}\right]_{z=tx}$$

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# CHAPTER X

## DYNAMICAL SYMMETRY ALGEBRA OF $F_2$

10.1) INTRODUCTION : Dynamical symmetry Algebra of Gaussian Hypergeometric function ${}_2F_1$ was constructed by Miller [1] and its further use was made by B.M. Agarwal and Renu Jain [4] to find generating functions for Jacobi Polynomials. In the present chapter we have introduced dynamical symmetry algebra of $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ and by its induced group action arrived at certain identities for $F_2$ which in their turn lead to reduction formulae for hypergeometric functions of three variable and generating functions for different polynomials.

10.2) The Dynamical Symmetry Algebra of $F_2$ : Let

$$f_{\alpha\beta\beta'\gamma\gamma'}(s,u,t,p,q,x,y)$$

$$= F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y)\, s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

be the basis elements of a subspace of analytical functions fo seven variables. $x, y, s, u, t, p, q$ associated with hypergeometric function $F_2$ defined as $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$.

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{i+j}\,(\beta)_i\,(\beta')_j}{(\gamma)_i\,(\gamma')_j} \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j}$$

The dynamical Symmetry algebra of $F_2$ is a complex Lie Algebra generated by E - operators termed as raising or lowering in view of their effect of raising or lowering the corresponding suffixin $f_{\alpha\beta\beta'\gamma\gamma'}$

$$f_{\alpha\beta\beta'\gamma\gamma'}(s,u,t,p,q,x,y) = K . F_2[\alpha,\beta,\beta',\gamma,\gamma',x,y] . s^{\alpha}u^{\beta}t^{\beta'}p^{\gamma}q^{\gamma'}$$

$$= K \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n}\frac{x^m}{\underline{|m}}\frac{y^n}{\underline{|n}} . s^{\alpha}u^{\beta}t^{\beta'}p^{\gamma}q^{\gamma'}$$

$$= K \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}} \sum_{m=0}^{\infty}\frac{(\alpha+n)_m(\beta)_m}{(\gamma)_m}\frac{x^m}{\underline{|m}} . s^{\alpha}u^{\beta}t^{\beta'}p^{\gamma}q^{\gamma'}$$

$$= K \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}} . {}_2F_1[\alpha+n,\beta;\gamma;x] . s^{\alpha}u^{\beta}t^{\beta'}p^{\gamma}q^{\gamma'}$$

or we can write it as

$$= K \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{x^m}{\underline{|m}} \sum_{n=0}^{\infty}\frac{(\alpha+m)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}$$

$$= K \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{x^m}{\underline{|m}} . {}_2F_1[\alpha+m,\beta';\gamma';y] . s^{\alpha}u^{\beta}t^{\beta'}p^{\gamma}q^{\gamma'}$$

The E - operators are

(i) $E_{\alpha} = s\left(x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}\right)$

(ii) $E_{-\alpha} = s^{-1}\left\{x(1-x)\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - s\frac{\partial}{\partial s}\right\}$

(iii) $E_{\beta} = u\left(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\right)$

(iv) $E_{-\beta} = u^{-1}\left\{x(1-x)\frac{\partial}{\partial x} - xs\frac{\partial}{\partial s} - u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}\right\}$

(v) $E_{\gamma} = p\left\{(1-x)\frac{\partial}{\partial x} - s\frac{\partial}{\partial s} - u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}\right\}$

(vi) $E_{-\gamma} = p^{-1}\left(x\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} - 1\right)$

(vii) $E_{\beta'} = t\left(y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right)$

(viii) $E_{-\beta'} = t^{-1}\left\{y(1-y)\frac{\partial}{\partial y} - ys\frac{\partial}{\partial s} - t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}\right\}$

(ix) $E_{\gamma'} = q\left\{(1-y)\frac{\partial}{\partial y} - s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}\right\}$

(x) $E_{-\gamma'} = q^{-1}\left[y\frac{\partial}{\partial y} + q\frac{\partial}{\partial q} - 1\right]$

(xi) $E_{\beta\gamma} = up\left[u\frac{\partial}{\partial u} - (1-x)\frac{\partial}{\partial x}\right]$

(xii) $E_{-\beta,-\gamma} = u^{-1}p^{-1}\left\{x(1-x)\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} - xs\frac{\partial}{\partial s} - 1\right\}$

(xiii) $E_{\alpha\gamma} = sp\left\{s\frac{\partial}{\partial s} - (1-x)\frac{\partial}{\partial x}\right\}$

(xiv) $E_{-\alpha,-\gamma} = s^{-1}p^{-1}\left\{x(1-x)\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - 1\right\}$

(xv) $E_{\beta',\gamma'} = tq\left\{t\frac{\partial}{\partial t} - (1-y)\frac{\partial}{\partial y}\right\}$

(xvi) $E_{-\beta',-\gamma'} = t^{-1}q^{-1}\left\{y(1-y)\frac{\partial}{\partial y} + q\frac{\partial}{\partial q} - ys\frac{\partial}{\partial s} - 1\right\}$

(xvii) $E_{\alpha\gamma'} = sq\left\{s\frac{\partial}{\partial s} - (1-y)\frac{\partial}{\partial y}\right\}$

(xviii) $E_{-\alpha,-\gamma'} = s^{-1}q^{-1}\left\{y(1-y)\frac{\partial}{\partial y} - yt\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} - 1\right\}$

(xix) $E_{\alpha\beta\gamma} = sup\frac{\partial}{\partial x}$

(xx) $E_{-\alpha,-\beta,-\gamma} = s^{-1}u^{-1}p^{-1}\left\{x(1-x)\frac{\partial}{\partial x} - xs\frac{\partial}{\partial s} - xu\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - 1\right\}$

(xxi) $E_{\alpha\beta'\gamma'} = stq\frac{\partial}{\partial y}$

(xxii) $E_{-\alpha,-\beta',-\gamma'} = s^{-1}t^{-1}q^{-1}\left\{y(1-y)\frac{\partial}{\partial y} - ys\frac{\partial}{\partial s} - yt\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} - 1\right\}$

The actions of these E-operators on $f_{\alpha\beta\beta'\gamma\gamma'}$ is given by

$$E_\alpha \cdot f_{\alpha\beta\beta'\gamma\gamma'}(s,u,t,p,q,x,y)$$

$$= s\left(x\frac{\partial}{\partial x}+s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}\right)\cdot f_{\alpha\beta\beta'\gamma\gamma'}(s,u,t,p,q,x,y)$$

$$= s\left(x\frac{\partial}{\partial x}+s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}\right)\left[K\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}\cdot\right.$$

$$\cdot {}_2F_1\{\alpha+n,\beta;\gamma;x\}\, s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}$$

$$= sK\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}\, s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}\cdot\right.$$

$$\cdot(\alpha+n)\,{}_2F_1\{\alpha+n+1,\beta;\gamma;x\}$$

$$-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}\, s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}$$

$$\cdot(\alpha+n)\,{}_2F_1\{\alpha+n,\beta;\gamma;x\}$$

$$+\alpha\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}\, s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}$$

$$\cdot {}_2F_1\{\alpha+n,\beta;\gamma;x\}$$

$$+\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{n y^n}{\underline{|n}}\, s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}$$

$$\cdot {}_2F_1\{\alpha+n,\beta;\gamma;x\}$$

used $\left(x\frac{\partial}{\partial x}+a\right){}_2F_1\{a,b;c;x\} = a\cdot{}_2F_1\{a+1,b;c;x\}$

$$= K\sum_{n=0}^{\infty}\frac{(\alpha+n)(\alpha)_n(\beta')_n}{(\gamma')_n}\frac{y^n}{\underline{|n}}\, s^{\alpha+1} u^\beta t^{\beta'} p^\gamma q^{\gamma'}.$$

$$\cdot {}_2F_1\{(\alpha+1)+n,\beta;\gamma;x\}$$

$$= \alpha\, f_{\alpha+1,\beta,\beta'\gamma\gamma'}(s,u,t,p,q,x,y)$$

Hence.

$$E_\alpha\cdot f_{\alpha\beta\beta'\gamma\gamma'}(s,u,t,p,q,x,y) = \alpha\cdot f_{\alpha+1,\beta,\beta'\gamma\gamma'}(s,u,t,p,q,x,y)$$

Similarly we can find action of other E-operators on

$f_{\alpha\beta\beta'\gamma\gamma'}$

$$E_{\pm\alpha}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\alpha\\ \gamma-\alpha\end{bmatrix} f_{\alpha\pm1,\beta,\beta'\gamma\gamma'}$$

$$E_{\pm\beta}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\beta\\ \gamma-\beta\end{bmatrix} f_{\alpha,\beta\pm1,\beta',\gamma,\gamma'}$$

$$E_{\pm\gamma}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma}\\ (\gamma-1)\end{bmatrix} f_{\alpha,\beta,\beta',\gamma\pm1,\gamma'}$$

$$E_{\pm\beta'}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\beta'\\ \gamma'-\beta'\end{bmatrix} f_{\alpha,\beta,\beta'\pm1,\gamma,\gamma'}$$

$$E_{\pm\gamma'}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{(\gamma'-\alpha)(\gamma'-\beta')}{\gamma'}\\ (\gamma'-1)\end{bmatrix} f_{\alpha\beta\beta'\gamma,\gamma'\pm1}$$

$$E_{\pm\beta,\pm\gamma}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\beta(\gamma-\alpha)}{\gamma}\\ (\gamma-1)\end{bmatrix} f_{\alpha,\beta\pm1,\beta',\gamma\pm1,\gamma'}$$

$$E_{\pm\alpha,\pm\gamma}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\alpha(\gamma-\beta)}{\gamma}\\ (\gamma-1)\end{bmatrix} f_{\alpha\pm1,\beta,\beta',\gamma\pm1,\gamma'}$$

$$E_{\pm\beta',\pm\gamma'}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\beta'(\gamma'-\alpha)}{\gamma'}\\ (\gamma'-1)\end{bmatrix} f_{\alpha,\beta,\beta'\pm1,\gamma,\gamma'\pm1}$$

$$E_{\pm\alpha,\pm\gamma'}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\alpha(\gamma'-\beta')}{\gamma'}\\ (\gamma'-1)\end{bmatrix} f_{\alpha\pm1,\beta,\beta',\gamma,\gamma'\pm1}$$

$$E_{\pm\alpha,\pm\beta,\pm\gamma}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\alpha\beta}{\gamma}\\ (\gamma-1)\end{bmatrix} f_{\alpha\pm1,\beta\pm1,\beta',\gamma\pm1,\gamma'}$$

$$E_{\pm\alpha,\pm\beta',\pm\gamma'}\; f_{\alpha\beta\beta'\gamma\gamma'} = \begin{bmatrix}\frac{\alpha\beta'}{\gamma'}\\ (\gamma'-1)\end{bmatrix} f_{\alpha\pm1,\beta,\beta'\pm1,\gamma,\gamma'\pm1}$$

The upper factor in each bracket is to be associated with plus sign and the lower with minus sign. These E-operators together with 5 maintenance operators $J_\alpha$, $J_\beta$, $J_{\beta'}$, $J_\gamma$, $J_{\gamma'}$ and identity operator I form a basis, Here

$$J_\alpha = s\frac{\partial}{\partial s}, \quad J_\beta = u\frac{\partial}{\partial u}, \quad J_{\beta'} = t\frac{\partial}{\partial t}$$

$$J_\gamma = p\frac{\partial}{\partial p}, \quad J_{\gamma'} = q\frac{\partial}{\partial q}, \quad I = 1$$

$$J_\alpha f_{\alpha\beta\beta'\gamma\gamma'} = \alpha . f_{\alpha\beta\beta'\gamma\gamma'}$$

$$J_\beta f_{\alpha\beta\beta'\gamma\gamma'} = \beta . f_{\alpha\beta\beta'\gamma\gamma'}$$

$$J_{\beta'} f_{\alpha\beta\beta'\gamma\gamma'} = \beta' . f_{\alpha\beta\beta'\gamma\gamma'}$$

$$J_\gamma f_{\alpha\beta\beta'\gamma\gamma'} = \gamma . f_{\alpha\beta\beta'\gamma\gamma'}$$

$$J_{\gamma'} f_{\alpha\beta\beta'\gamma\gamma'} = \gamma' . f_{\alpha\beta\beta'\gamma\gamma'}$$

Section (1)

10.3) In the present section a group theoretic basis has been provided to derive reduction formulae for hypergeometric functions in three variables First of all we employ the operator

10.3.1) $$E_{\alpha\gamma} = sp\left\{s\frac{\partial}{\partial s} - (1-x)\frac{\partial}{\partial x}\right\}$$

with action

10.3.2) $$E_{\alpha\gamma} f_{\alpha\beta\beta'\gamma\gamma'} = \frac{\alpha(\gamma-\beta)}{\gamma} f_{\alpha+1,\beta,\beta',\gamma+1,\gamma'}$$

To find $\exp a E_{\alpha\gamma}$ we use the standard Lie theoretic technique [2]. It can be computed

by solving the differential equations.

$$\frac{d\,S(a)}{da} = S^2(a).\,p. \quad , \quad S(0) = S$$

or. $$\int \frac{d\,S(a)}{S^2(a)} = \int p.\,da. + K$$

or, $$-\frac{1}{S(a)} = a.\,p + K$$

when $a = 0$, $s(0) = s$ then $K = -\frac{1}{S}$ and.

$$S(a) = \frac{S}{1-paS}$$

and

$$\frac{d\,x(a)}{da} = \{x(a)-1\}\,S(a).\,p.$$

$$\int \frac{d\,x(a)}{\{x(a)-1\}} = \int \frac{Sp}{1-paS}\,da. + K.$$

$$\log\{x(a)-1\} = -\log(1-paS) + K$$

When $a = 0$ $x(0) = x$ then $K = \log(x-1)$

$$x(a) = \frac{x-paS}{1-paS}$$

So that

10.3.3) $$(\exp a E_{\alpha\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left[\alpha,\beta,\beta',\gamma,\gamma', \frac{x-paS}{1-paS}, y\right].$$
$$\cdot\left(\frac{S}{1-paS}\right)^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

On the other hand by direct expansion we have

10.3.4) $$(\exp a E_{\alpha\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n}(E_{\alpha\gamma})^n f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\alpha)_n(\gamma-\beta)_n}{(\gamma)_n} f_{\alpha+n,\beta,\beta',\gamma+n,\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\alpha)_n(\gamma-\beta)_n}{(\gamma)_n}.$$

$$\cdot F_2(\alpha+n,\beta,\beta',\gamma+n,\gamma',x,y).\,S^{\alpha+n} u^{\beta} t^{\beta'} p^{\gamma+n} q^{\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma-\beta)_n}{(\gamma)_n} S^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'} \frac{(aSp)^n}{\lfloor n}.$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha+n)_{i+j}\,(\beta)_i\,(\beta')_j}{(\gamma+n)_i\,(\gamma')_j}\,\frac{x^i}{\underline{|i}}\,\frac{y^j}{\underline{|j}}$$

Equating the two values of exp $(a\,E_{\alpha\gamma})$ we get

10.3.5) $$F_2\left[\alpha,\beta,\beta',\gamma,\gamma',\frac{x-pas}{1-pas},y\right](1-pas)^{-\alpha}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_n\,(\alpha+n)_{i+j}\,(\gamma-\beta)_n\,(\beta)_i\,(\beta')_j}{(\gamma)_n\,(\gamma+n)_i\,(\gamma')_j}\cdot\frac{x^i}{\underline{|i}}\,\frac{y^j}{\underline{|j}}\,\frac{(asp)^n}{\underline{|n}}$$

which finally gives

10.3.6) $$(1-pas)^{-\alpha}\cdot F_2\left(\alpha,\beta,\beta',\gamma,\gamma',\frac{x-pas}{1-pas},y\right)$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{n+i+j}\,(\gamma-\beta)_n\,(\beta)_i\,(\beta')_j}{(\gamma)_{n+i}\,(\gamma')_j}\cdot\frac{x^i}{\underline{|i}}\,\frac{y^j}{\underline{|j}}\,\frac{(asp)^n}{\underline{|n}}$$

setting $asp\to z$, $\beta'=\gamma'$, $(\gamma-\beta)\to\beta_1$

and in view of definition [3]

10.3.7) $$F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\beta_1,\gamma_2,\gamma_2;x,y,z)$$
$$=(1-x)^{-\alpha_1}\cdot F_1\left(\alpha_1,\beta_2,\beta_3;\gamma_2;\frac{y}{1-x},\frac{z}{1-x}\right)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(\alpha_1)_{m+n+i}\,(\beta_2)_m\,(\beta_3)_n}{(\gamma_2)_{m+n}}\cdot\frac{x^i}{\underline{|i}}\,\frac{y^m}{\underline{|m}}\,\frac{z^n}{\underline{|n}}$$

we arrive at the reduction formula.

10.3.8) $$(1-z)^{-\alpha}\cdot F_2\left(\alpha,\beta,\beta';\beta+\beta_1,\gamma',\frac{x-z}{1-z},y\right)$$
$$=F_G(\alpha,\alpha,\alpha,\beta',\beta_1,\beta;\beta',\beta+\beta_1,\beta+\beta_1;x,y,z)$$

10.4) Next we use the operator :-

10.4.1) $$E_\beta=u\left(x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}\right)$$

with action

10.4.2) $$E_\beta\,f_{\alpha\beta\beta'\gamma\gamma'}=\beta\cdot f_{\alpha,\beta+1,\beta'\gamma\gamma'}$$

To find $(\exp aE_\beta)$ we use the standard Lie theoretic technique [2]. It can be computed by solving

following differential equation.

(i) $$\frac{du(a)}{da} = u^2(a) \quad , \quad u(0) = u$$

or $$\int \frac{du(a)}{u^2(a)} = \int da + K$$

or $$-\frac{1}{u(a)} = a + K$$

when $a = 0$ $\quad u(0) = u \quad$ then $K = -\frac{1}{u}$

$$u(a) = \frac{u}{1-au}$$

and

(ii) $$\frac{d x(a)}{da} = u(a) \, . \, x(a)$$

$$\frac{d x(a)}{x(a)} = \frac{u}{1-au} \, . \, da$$

$$\log x(a) = -\log(1-au) + K$$

when $a = 0$ $\quad K = \log x$

$$\log x(a) = \log x - \log(1-au)$$

$$x(a) = \frac{x}{1-au}$$

which gives

(10.4.3) $$(\exp a E_\beta) f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-au}, y\right) . \frac{s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}}{(1-au)^\beta}.$$

on the other hand by direct expansion, we get

$$(\exp a E_\beta) f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (E_\beta)^n f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (\beta)_n \, f_{\alpha, \beta+n, \beta', \gamma, \gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (\beta)_n \, F_2(\alpha, \beta+n, \beta', \gamma, \gamma', x, y) . s^\alpha u^{\beta+n} t^{\beta'} p^\gamma q^{\gamma'}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a^n}{\lfloor n} (\beta)_n \, s^\alpha u^{\beta+n} t^{\beta'} p^\gamma q^{\gamma'}.$$

$$\cdot \frac{(\alpha)_{i+j} (\beta+n)_i \, (\beta')_j}{(\gamma)_i \, (\gamma')_j} \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j}$$

10.4.4) $$(\exp a E_\beta) f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{i+j} (\beta)_{i+n} (\beta')_j}{(\gamma)_i \, (\gamma')_j} \cdot \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j} \frac{(au)^n}{\lfloor n} \cdot s^\alpha u^\beta t^{\beta'} p^\gamma q^{\gamma'}$$

Equating the two values of $(\exp a E_\beta) f_{\alpha\beta\beta'\gamma\gamma'}$ we get the identity

10.4.5) $$(1-au)^{-\beta} \cdot F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-au}, y\right) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{i+j} (\beta)_{i+n} (\beta')_j}{(\gamma)_i \, (\gamma')_j} \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j} \frac{(au)^n}{\lfloor n}$$

setting $au \to z$ and using the definition [3]

10.4.6) $$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_3, x, y, z) = (1-x)^{-\beta_1} \cdot F_2\left(\alpha_2, \beta_2, \beta_1, \gamma_2, \gamma_3, y, \frac{z}{1-x}\right)$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{(\alpha_2)_{m+n} (\beta_2)_m (\beta_1)_{n+i}}{(\gamma_2)_m (\gamma_3)_n} \cdot \frac{x^i}{\lfloor i} \cdot \frac{y^m}{\lfloor m} \frac{z^n}{\lfloor n}$$

we arrive at the reduction formula.

10.4.7) $$(1-z)^{-\beta_1} \cdot F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-z}, y\right) = F_K(\alpha', \alpha, \alpha, \beta, \beta', \beta; \alpha', \gamma, \gamma', x, y, z)$$

10.5) Again we employ the operator

10.5.1) $$E_{\beta\gamma} = up\left(u\frac{\partial}{\partial u} - (1-x)\frac{\partial}{\partial x}\right)$$

with action

10.5.2) $$E_{\beta\gamma} f_{\alpha\beta\beta'\gamma\gamma'} = \frac{\beta(\gamma-\alpha)}{\gamma} f_{\alpha, \beta+1, \beta', \gamma+1, \gamma'}$$

To find $(\exp a E_{\beta\gamma})$ we use the standard Lie theoretic technique [2]. It can be computed by solving following equations.

(i) $$\frac{du(a)}{da} = u^2(a) \cdot p$$

10.5.6) $F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_2; x, y, z)$
$= (1-x)^{-\beta_1} \cdot F_1\left(\alpha_2, \beta_2, \beta_1; \gamma_2; y, \frac{z}{1-x}\right)$

It takes the form of the reduction formula

10.5.7) $(1-z)^{-\beta} \cdot f_2\left(\alpha, \beta, \beta', \alpha+\beta_1, \beta'; \frac{x-z}{1-z}, y\right)$
$= F_M(\alpha_1, \alpha, \alpha, \beta_1, \beta, \beta_1, \alpha_1, \alpha+\beta_1, \alpha+\beta_1, x, y, z)$

10.6) Now we employ the operator

10.6.1) $E_\alpha = s\left(x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y}\right)$

with action

10.6.2) $E_\alpha f_{\alpha\beta\beta'\gamma\gamma'} = \alpha \cdot f_{\alpha+1, \beta, \beta', \gamma, \gamma'}$

for computing action of one parameter subgroup $(\exp a E_\alpha)$ by usual multiplier representation theory we have to solve following differential equations

(i) $\frac{d\,s(a)}{da} = s^2(a)$

$$\int \frac{d\,s(a)}{s^2(a)} = \int da + K$$

$$-\frac{1}{s(a)} = a + K$$

when $a = 0$, $s(0) = s$ then $K = -\frac{1}{s}$

$$s(a) = \frac{s}{1-as}$$

(ii) $\frac{d\,x(a)}{da} = s(a) \cdot x(a)$

$$\int \frac{d\,x(a)}{x(a)} = \int \frac{s \cdot da}{1-as} + K$$

$$\log x(a) = -\log(1-as) + K$$

when $a = 0$, $s(0) = s$, then $K = \log x$

$$x(a) = \frac{x}{1-as}$$

(iii) $\frac{d\,y(a)}{da} = s(a) \cdot y(a)$

$$-\frac{1}{u(a)} = pa + K$$

When $a = 0$, $u(o) = u$ then $K = -\frac{1}{u}$

$$u(a) = \frac{u}{1-pau}$$

(ii)
$$\frac{d\,x(a)}{da} = \{x(a)-1\}\,u(a)\,.\,p.$$

$$\int \frac{d\,x(a)}{\{x(a)-1\}} = \int \frac{u\,p}{1-pau}\,.\,da + K$$

$$\operatorname{Log}\{x(a)-1\} = -\log(1-pau) + K$$

when $a = 0$, $x(o)=x$ then $K = \operatorname{Log}(x - 1)$

$$x(a) = \frac{x-pau}{1-pau}$$

Thus

10.5.3)
$$(\exp a E_{\beta\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{x-pau}{1-pau}, y\right] . \frac{s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}}{(1-pau)^{\beta}}$$

On the other hand by direct expansion we get

$$(\exp a E_{\beta\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{n=0}^{\infty} \frac{a^n\,(E_{\beta\gamma})^n}{\lfloor n} f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\beta)_n\,(\gamma-\alpha)_n}{(\gamma)_n} f_{\alpha,\beta+n,\beta',\gamma+n,\gamma'}$$

10.5.4)
$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\beta)_n\,(\gamma-\alpha)_n}{(\gamma)_n} F_2(\alpha, \beta+n, \beta', \gamma+n, \gamma', x, y) . s^{\alpha} u^{\beta+n} t^{\beta'} p^{\gamma+n} q^{\gamma'}$$

Equating two values of $(\exp a E_{\beta\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$ we arrive at the identity

10.5.5)
$$(1-pau)^{-\beta} . F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{x-pau}{1-pau}, y\right]$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\beta)_{i+n}(\gamma-\alpha)_n}{(\gamma)_{i+n}} . \frac{(\alpha)_{i+j}\,(\beta')_j}{(\gamma')_j} . \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j} \frac{(aup)^n}{\lfloor n}$$

Setting $aup \to z$, $(\gamma-\alpha) \to \beta_1$, $\beta' \to \gamma'$ interchanging $\alpha$ and $\beta$ and using the definition

$$\int \frac{dy(a)}{y(a)} = \int \frac{s}{1-as}\, da + K$$

$$\log y(a) = -\log(1-as) + K$$

when $a = 0$ $y(0) = y$ then $K = \log y$

$$y(a) = \frac{y}{1-as}$$

So that

10.6.3) $$(\exp aE_\alpha)\, f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-as}, \frac{y}{1-as}\right) \cdot \left(\frac{s}{1-as}\right)^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

On the other hand by direct expansion

$$(\exp aE_\alpha)\, f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{m=0}^{\infty} \frac{a^m (E_\alpha)^m}{\lfloor m} f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} (\alpha)_m\, f_{\alpha+m, \beta, \beta'\gamma\gamma'}$$

10.6.4) $$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} (\alpha)_m \cdot F_2(\alpha+m, \beta, \beta', \gamma, \gamma', x, y) \cdot s^{\alpha+m} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

By comparing the two values we get

$$(1-as)^{-\alpha} \cdot F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-as}, \frac{y}{1-as}\right) \cdot s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a^m (\alpha)_m}{\lfloor m} s^{\alpha+m} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'} \cdot \frac{(\alpha+m)_{i+j} (\beta)_i (\beta')_j}{(\gamma)_i (\gamma')_j} \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j}$$

which after simplification gives

10.6.5) $$(1-as)^{-\alpha} \cdot F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-as}, \frac{y}{1-as}\right)$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{m+i+j} (\beta)_i (\beta')_j}{(\gamma)_i (\gamma')_j} \cdot$$

$$\cdot \frac{x^i}{\lfloor i} \frac{y^j}{\lfloor j} \frac{(as)^m}{\lfloor m}$$

Setting as → Z and in view of the definition.

$$10.6.6)\ F_C\left[\alpha, \beta_1, \beta_2, \beta_3;\ \gamma_1, \gamma_2, \beta_3;\ x, y, z\right]$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta_1)_m (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_n} \cdot \frac{x^m}{\lfloor m} \frac{y^n}{\lfloor n} \frac{z^p}{\lfloor p}.$$

We arrive at the reduction formula.

$$10.6.7)\ (1-z)^{-\alpha} \cdot F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{x}{1-z}, \frac{y}{1-z}\right]$$

$$= F_C\left[\alpha, \beta, \beta', \beta^*;\ \gamma, \gamma', \beta^*;\ x, y, z\right]$$

<u>Section [2]</u>

10.7) In the present section a group theoretic basis has been provided to obtain generating functions, first of all we employ the operator.

$$10.7.1)\ E_{-\alpha,-\gamma} = s^{-1} p^{-1}\left[x(1-x)\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - 1\right]$$

with action

$$10.7.2)\ E_{-\alpha,-\gamma}\, f_{\alpha\beta\beta'\gamma\gamma'} = (\gamma-1)\, f_{\alpha-1,\beta,\beta',\gamma-1,\gamma'}$$

To find $(\exp a E_{-\alpha,-\gamma})$ we use the standard Lie theoretic technique [2]. It can be computed by solving the equations.

(i) $$\frac{d\,p(a)}{da} = \frac{1}{s}$$

$$\int d\,p(a) = \frac{1}{s}\int da + K$$

$$p(a) = \frac{a}{s} + K$$

when $a = 0$ $\quad p(a) = 0 \quad K = p$

$$p(a) = \frac{a}{s} + p.$$

(ii) $$\frac{dx(a)}{da} = \frac{x(a)\{1 - x(a)\}}{s.\,p(a)}$$

$$\frac{dx(a)}{x(a)\{1-x(a)\}} = \frac{da}{a+sp}$$

$$\int \frac{dx(a)}{x(a)} + \int \frac{dx(a)}{1-x(a)} = \int \frac{da}{a+sp} + K$$

$$\log x(a) - \log\{1-x(a)\} = \log(a+sp) + K$$

when $a = 0$, $\quad x(0) = x \quad$ then $K = \log \frac{x}{sp(1-x)}$

$$x(a) = \frac{x(a+ps)}{ps + ax}$$

(iii) $$\frac{du(a)}{da} = -\frac{x(a)\,u(a)}{s.\,p(a)}$$

$$\frac{du(a)}{u(a)} = -\frac{x(a+ps)}{(ps+ax)(a+ps)}.\,da.$$

$$\log u(a) = -\log(ax+ps) + K$$

when $a = 0$, $\quad u(0) = u \quad$ then $K = \log ups.$

$$u(a) = \frac{ups}{ax+ps}$$

(iv) $$\frac{dv(a)}{da} = -\frac{1}{sp(a)} \cdot v(a)$$

$$\int \frac{dv(a)}{v(a)} = -\int \frac{da}{a+ps} + K$$

$$\log v(a) = -\log(a+ps) + K$$

when $a = 0$, $\quad v(0) = 1 \quad$ then $K =$ ~~$\log ps$~~ $\log ps$

$$v(a) := \frac{ps}{a+ps}$$

Thus

10.7.3) $(\exp a E_{-\alpha,-\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$

$$= \frac{sp}{a+sp} F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{x(a+ps)}{ps+ax}, y\right] \cdot$$

$$\cdot s^{\alpha}\left(\frac{ups}{ax+ps}\right)^{\beta} t^{\beta'}\left(\frac{a+sp}{s}\right)^{\gamma} q^{\gamma'}$$

$$= F_2\left[\alpha, \beta, \beta', \gamma, \gamma'; \frac{x(a+ps)}{ps+ax}, y\right] \cdot$$

$$\cdot s^{\alpha+1-\gamma+\beta} p^{\beta+1} u^{\beta} t^{\beta'} q^{\gamma'} (a+sp)^{\gamma-1} (ax+ps)^{-\beta}$$

On the other hand by direct expansion it gives

$(\exp a E_{-\alpha,-\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$

10.7.4) $$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (E_{-\alpha,-\gamma})^n f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (\gamma-n)_n f_{\alpha-n,\beta,\beta',\gamma-n,\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (\gamma-n)_n \cdot$$

$$\cdot F_2(\alpha-n, \beta, \beta', \gamma-n, \gamma'; x, y)$$

$$\cdot s^{\alpha-n} u^{\beta} t^{\beta'} p^{\gamma-n} q^{\gamma'}$$

Equating the two values of $(\exp a E_{-\alpha,-\gamma}) \cdot$ $\cdot f_{\alpha\beta\beta'\gamma\gamma'}$ we get

10.7.5) $$F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{x(a+ps)}{ps+ax}, y\right] \cdot$$

$$\cdot s^{\alpha+\beta-\gamma+1} p^{\beta+1} u^{\beta} t^{\beta'} q^{\gamma'} (a+sp)^{\gamma-1} (ax+ps)^{-\beta}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (\gamma-n)_n \cdot F_2\left[\alpha-n, \beta, \beta', \gamma-n, \gamma', x, y\right] \cdot$$

$$\cdot s^{\alpha-n} u^{\beta} t^{\beta'} p^{\gamma-n} q^{\gamma'}$$

which finally gives a generating relation as

10.7.6) $F_2\left[\alpha, \beta, \beta', \gamma, \gamma'; \frac{x(a+ps)}{ps+ax}, y\right] \cdot$

$$\cdot\, s^{\beta-\gamma+1}\, p^{\beta+1-\gamma}\, (a+sp)^{\gamma-1}\, (ax+ps)^{-\beta}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n}\, (\gamma-n)_n\, (sp)^{-n} \cdot$$

$$\cdot\, F_2\left[\alpha-n, \beta, \beta', \gamma-n, \gamma', x, y\right]$$

10.8) Next we use the operator

10.8.1) $E_{-\alpha} = s^{-1}\left[x(1-x)\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - s\frac{\partial}{\partial s}\right]$

with action

10.8.2) $E_{-\alpha}\, f_{\alpha\beta\beta'\gamma\gamma'} = (\gamma-\alpha)\, f_{\alpha-1,\beta\beta'\gamma\gamma'}$

The action of one parameter subgroup $(\exp a E_{-\alpha})$ is found by standard Lie - theoretic technique [2]

Its computed by solving the differential equations.

(i) $$\frac{d\,S(a)}{da} = -1$$

$$\int dS(a) = -\int da + K$$

$$S(a) = -a + K$$

when $a = 0$, $s(0) = s$ then $K = s$

$$S(a) = S - a$$

(ii) $$\frac{dp(a)}{da} = \frac{p(a)}{S(a)}$$

$$\int \frac{dp(a)}{p(a)} = \int \frac{da}{S-a} + K$$

$$\log p(a) = -\log(S-a) + K$$

when $a = 0$, $p(0) = p$ then $K = \log sp$

$$p(a) = \frac{sp}{s-a}$$

(iii) $$\frac{dx(a)}{da} = \frac{x(a)\{1-x(a)\}}{s(a)}$$

$$\int \frac{dx(a)}{x(a)\{1-x(a)\}} = \int \frac{da}{s-a} + K$$

$$\int \frac{dx(a)}{x(a)} + \int \frac{dx(a)}{1-x(a)} = \int \frac{da}{s-a} + K$$

$$\log x(a) - \log\{1-x(a)\} = -\log(s-a) + K$$

when $a = 0$, $x(0) = x$ $K = \log \dfrac{xs}{1-x}$

$$\frac{x(a)}{1-x(a)} = \frac{xs}{(1-x)(s-a)}$$

$$x(a) = \frac{xs}{a(x-1)+s}$$

(iv) $$\frac{du(a)}{da} = -\frac{x(a)\,u(a)}{s(a)}$$

$$\frac{du(a)}{u(a)} = -\frac{xs\,.\,da}{\{a(x-1)+s\}(s-a)}$$

$$u(a) = \frac{u(s-a)}{a(x-1)+s}$$

so that

10.8.3) $$(\exp aE_{-\alpha})\, f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{xs}{a(x-1)+s}, y\right].$$

$$\cdot (s-a)^{\alpha} \frac{u^{\beta}(s-a)^{\beta}}{\{a(x-1)+s\}^{\beta}} t^{\beta'} \frac{(ps)^{\gamma}}{(s-a)^{\gamma}} q^{\gamma'}$$

On the other hand by direct expansion it gives.

10.8.4) $$(\exp a E_{-\alpha}) f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} (E_{-\alpha})^m f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} (\gamma-\alpha)_m \cdot F_2[\alpha-m, \beta, \beta', \gamma, \gamma', x, y].$$

$$\cdot s^{\alpha-m} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

Equating the two values of $(\exp a E_{-\alpha}) f_{\alpha\beta\beta'\gamma\gamma'}$

we get the identity

10.8.5) $$F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{xs}{a(x-1)+s}, y\right] . \{a(x-1)+s\}^{-\beta}.$$

$$\cdot (s-a)^{\alpha+\beta-\gamma} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'} s^{\gamma}$$

$$= \sum_{m=0}^{\infty} \frac{(s^{-1}a)^m}{\lfloor m} (\gamma-\alpha)_m s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}.$$

$$\cdot F_2[\alpha-m, \beta, \beta', \gamma, \gamma', x, y]$$

finally

which gives a generating relation.

10.8.6) $$\{a(x-1)+s\}^{-\beta} (s-a)^{\alpha+\beta-\gamma}.$$

$$\cdot F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{xs}{a(x-1)+s}, y\right]$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} (\gamma-\alpha)_m s^{\alpha-m-\gamma}.$$

$$\cdot F_2[\alpha-m, \beta, \beta', \gamma, \gamma', x, y]$$

10.9) Now we employ the operator

10.9.1) $$E_{-\beta} = u^{-1}\left[x(1-x)\frac{\partial}{\partial x} - xs\frac{\partial}{\partial s} - u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}\right]$$

with action

10.9.2) $$E_{-\beta} f_{\alpha\beta\beta'\gamma\gamma'} = (\gamma-\beta) f_{\alpha, \beta-1, \beta', \gamma\gamma'}$$

Computing action of one parameter subgroup $(\exp a E_{-\beta})$ by usual multiplier representation theory we solve these differential equations.

(i)
$$\frac{du(a)}{da} = -1$$

$$\int du(a) = -\int da + K$$

$$u(a) = -a + K$$

when $a = 0$, $u(o) = u$, $K = u$

$$u(a) = u - a$$

(ii)
$$\frac{dp(a)}{da} = \frac{p(a)}{u(a)}$$

$$\int \frac{dp(a)}{p(a)} = \int \frac{da}{u-a} + K$$

$$\log p(a) = -\log(u-a) + K$$

when $a = 0$, $p(o) = p$ then $K = \log pu$

$$p(a) = \frac{pu}{u-a}$$

(iii)
$$\frac{dx(a)}{da} = \frac{x(a)\{1 - x(a)\}}{u(a)}$$

$$\int \frac{dx(a)}{x(a)\{1 - x(a)\}} = \int \frac{da}{u-a} + K$$

$$\int \frac{dx(a)}{x(a)} + \int \frac{dx(a)}{1 - x(a)} = \int \frac{da}{u-a} + K$$

$$\log x(a) - \log\{1-x(a)\} = -\log(u-a)+K$$

when a = 0, $x(0) = x$ then $K = \log \frac{xu}{1-x}$

$$\frac{x(a)}{1-x(a)} = \frac{xu}{(1-x)(u-a)}$$

$$x(a) = \frac{xu}{u-a(1-x)}$$

(iv) $$\frac{dS(a)}{da} = -\frac{x(a).S(a)}{u(a)}$$

$$\frac{dS(a)}{S(a)} = -\frac{xu}{u-a(1-x)} \cdot \frac{da}{(u-a)}$$

$$S(a) = \frac{S(u-a)}{a(x-1)+u}$$

Thus

10.9.3) $$(\exp a E_{-\beta}) f_{\alpha\beta\beta'\gamma\gamma'} = F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{ux}{a(x-1)+u}, y\right). \frac{S^\alpha (u-a)^\alpha}{\{a(x-1)+u\}^\alpha} \cdot (u-a)^\beta t^{\beta'} \frac{(pu)^\gamma}{(u-a)^\gamma} q^{\gamma'}$$

$$= F_2\left(\alpha, \beta, \beta', \gamma, \gamma', \frac{ux}{a(x-1)+u}, y\right). S^\alpha (u-a)^{\alpha+\beta-\gamma} p^\gamma q^{\gamma'} u^\gamma t^{\beta'}. \{a(x-1)+u\}^{-\alpha}$$

On the other hand by direct expansion it yields.

10.9.4) $$(\exp a E_{-\beta}) f_{\alpha\beta\beta'\gamma\gamma'} = \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (E_{-\beta})^n f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (\gamma-\beta)_n \, f_{\alpha;\beta-n,\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\underline{|n}} (\gamma-\beta)_n \cdot F_2(\alpha, \beta-n, \beta', \gamma, \gamma', x, y). S^\alpha u^{\beta-n} t^{\beta'} p^\gamma q^{\gamma'}$$

Equating the two values of $(\exp a E_{-\beta}) f_{\alpha\beta\beta'\gamma\gamma'}$

10.9.5) $\{a(x-1)+u\}^{-\alpha}\,(u-a)^{\alpha+\beta-\gamma}\; u^{\gamma-\beta}\,.$

$$\cdot\, F_2\left[\alpha, \beta, \beta', \gamma, \gamma', \frac{ux}{a(x-1)+u}, y\right]$$

$$= \sum_{n=0}^{\infty} (\gamma-\beta)_n \,.\, F_2(\alpha, \beta-n, \beta', \gamma, \gamma', x, y).$$

$$\cdot\,\frac{(au^{-1})^n}{\lfloor n}$$

10.10) Now we use the operator

10.10.1) $E_\gamma = p\left[(1-x)\frac{\partial}{\partial x} - s\frac{\partial}{\partial s} - u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}\right]$

with action

10.10.2) $E_\gamma\, f_{\alpha\beta\beta'\gamma\gamma'} = \dfrac{(\gamma-\alpha)(\gamma-\beta)}{\gamma}\, f_{\alpha,\beta,\beta',\gamma+1,\gamma'}$

Computing action of one parameter subgroup ( exp a $E_\gamma$ ) by usual multiplier representation. theory

(i) $$\frac{d\,p(a)}{da} = p^2(a)$$

$$\int \frac{d\,p(a)}{p^2(a)} = \int da + K$$

$$-\frac{1}{p(a)} = a + K$$

when $a = 0$, $p(0) = p$ then $K = -\frac{1}{p}$

$$p(a) = \frac{p}{1-ap}$$

(ii) $$\frac{du(a)}{da} = -u(a).\,p(a)$$

$$\int \frac{du(a)}{u(a)} = -\int \frac{p}{1-ap}\,da + K$$

$$\log u(a) = \log(1-ap) + K$$

when $a = 0$, $u(0) = u$ then $K = \log u$

$$\log u(a) = \log(1-ap) + \log u$$

$$u(a) = u(1-ap)$$

(iii) $$\frac{d\,S(a)}{da} = -S(a).\,p(a)$$

$$\int \frac{d\,S(a)}{S(a)} = -\int \frac{p.\,da}{1-ap} + K$$

$$\log S(a) = \log(1-ap) + K$$

when $a = 0$, $s(0) = s$ then $K = \log s$

$$S(a) = S(1-ap)$$

(iv) $$\frac{d\,x(a)}{da} = \{1-x(a)\}.\,p(a)$$

$$\int \frac{d\,x(a)}{1-x(a)} = \int \frac{p.\,da}{1-ap} + K$$

$$-\log\{1-x(a)\} = -\log(1-ap) + K$$

when $a = 0$, $x(0) = x$, then $K = -\log(1 - x)$

$$1 - x(a) = (1-ap).\,(1-x)$$

$$x(a) = x(1-ap) + ap$$

Therefore

10.10.3) $$(\exp a E_r)\, f_{\alpha\beta\beta' r r'}$$
$$= F_2(\alpha, \beta, \beta', r, r', x(1-ap)+ap, y).$$
$$.\, s^{\alpha}(1-ap)^{\alpha}\, u^{\beta}(1-ap)^{\beta+\beta'}\, p^{r}(1-ap)^{-r}.$$
$$.\, q^{r'}$$

On the other hand by direct expansion it yields

10.10.4) $(\exp a E_\gamma) f_{\alpha\beta\beta'\gamma\gamma'}$

$$= \sum_{m=0}^{\infty} \frac{a^m (E_\gamma)^m}{\lfloor m} f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{\lfloor m} \frac{(\gamma-\alpha)_m (\gamma-\beta)_m}{(\gamma)_m} .$$

$$\cdot F_2(\alpha, \beta, \beta', \gamma+m, \gamma', x, y) \cdot s^\alpha u^\beta t^{\beta'} p^{\gamma+m} q^{\gamma'}$$

Equating two values of $(\exp a E_\gamma) f_{\alpha\beta\beta'\gamma\gamma'}$

we arrive at the generating relation.

10.10.5) $F_2(\alpha, \beta, \beta', \gamma, \gamma', x(1-ap)+ap, y) \cdot (1-ap)^{\alpha+\beta-\gamma}$

$$= \sum_{m=0}^{\infty} \frac{(ap)^m}{\lfloor m} \frac{(\gamma-\alpha)_m (\gamma-\beta)_m}{(\gamma)_m} .$$

$$\cdot F_2[\alpha, \beta, \beta', \gamma+m, \gamma', x, y]$$

10.11) Next we use the operator

10.11.1) $E_{-\gamma} = p^{-1}\left(x\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} - 1\right)$

with action

10.11.2) $E_{-\gamma} f_{\alpha\beta\beta'\gamma\gamma'} = (\gamma-1) f_{\alpha\beta\beta'(\gamma-1)\gamma'}$

Computing action of one parameter subgroup by usual multiplier representation theory we have to solve these differential equations.

(i) $$\frac{dp(a)}{da} = 1.$$

$$\int dp(a) = \int da + K$$

$$p(a) = a + K$$

When $a = 0$, $p(0) = p$, then $K = p$

$$p(a) = a + p$$

(ii)
$$\frac{d\,x(a)}{da} = \frac{x(a)}{p(a)}$$

$$\int \frac{d\,x(a)}{x(a)} = \int \frac{da}{p+a}$$

$$\log x(a) = \log(p+a) + K$$

when $a = 0$, $x(0) = x$, then $K = \log \frac{x}{p}$

$$x(a) = \frac{x}{p}(a+p)$$

(iii)
$$\frac{d\,v(a)}{da} = -\frac{1}{p(a)} \cdot v(a)$$

$$\int \frac{d\,v(a)}{v(a)} = -\int \frac{da}{a+p} + K$$

$$\log v(a) = -\log(a+p) + K$$

when $a = 0$, $v(0) = 1$ then $K = \log p$.

$$v(a) = \frac{p}{a+p}.$$

So that

10.11.3) $$(\exp a E_{-r})\, f_{\alpha\beta\beta' r r'}$$
$$= p \cdot F_2\left(\alpha, \beta, \beta', r, r', \frac{x}{p}(a+p), y\right),$$
$$\cdot\, s^{\alpha} u^{\beta} t^{\beta'} (a+p)^{r-1} q^{r'}$$

On the other hand by direct ~~expansion it yields~~ expansion it yields

10.11.4) $$(\exp aE_{-\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n (E_{-\gamma})^n}{\lfloor n} f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (\gamma-n)_n\, f_{\alpha\beta\beta'(\gamma-n)\gamma'}$$

Equating two values of $(\exp aE_{-\gamma})\, f_{\alpha\beta\beta'\gamma\gamma'}$ we arrive at the identity

$$p.\; F_2\left(\alpha,\beta,\beta',\gamma,\gamma', \frac{x}{p}(a+p), y\right).$$
$$\cdot s^{\alpha} u^{\beta} t^{\beta'} (a+p)^{\gamma-1} q^{\gamma'}$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (\gamma-n)_n \cdot F_2(\alpha,\beta,\beta',\gamma-n,\gamma',x,y).$$
$$\cdot s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma-n} q^{\gamma'}$$

which finally gives the generating relation

10.11.5) $$\left(1+\frac{a}{p}\right)^{\gamma-1} \cdot F_2\left(\alpha,\beta,\beta',\gamma,\gamma', \frac{x}{p}(a+p), y\right)$$
$$= \sum_{n=0}^{\infty} \frac{(ap^{-1})^n}{\lfloor n} (\gamma-n)_n \cdot F_2(\alpha,\beta,\beta',\gamma-n,\gamma',x,y)$$

10.12) Lastly we use the operator

10.12.1) $$E_{\alpha\beta\gamma} = s u p \frac{\partial}{\partial x}$$

with action

10.12.2) $$E_{\alpha\beta\gamma}\, f_{\alpha\beta\beta'\gamma\gamma'} = \frac{\alpha\beta}{\gamma} f_{\alpha+1,\beta+1,\beta',\gamma+1,\gamma'}$$

Computing action of one parameter subgroup $(\exp aE_{\alpha\beta\gamma})$ by usual multiplier representation

Theory we solve these equations

$$\frac{d\mathfrak{x}(a)}{da} = \text{sup}.$$

$$\int d\mathfrak{x}(a) = \int \text{sup}.\, da + K$$

$$\mathfrak{x}(a) = \text{sup}\, a + K$$

when $a = 0$, $\mathfrak{x}(0) = x$ then $K = x$

$$\mathfrak{x}(a) = \text{sup}\, a + \mathfrak{x}$$

Thus,

10.12.3) $(\exp a E_{\alpha\beta\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$

$$= F_2(\alpha, \beta, \beta', \gamma, \gamma', \mathfrak{x} + \text{sup}\, a, y) \cdot s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

On the other hand by direct expansion we get

10.12.4) $(\exp a E_{\alpha\beta\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} (E_{\alpha\beta\gamma})^n f_{\alpha\beta\beta'\gamma\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot F_2(\alpha+n, \beta+n, \beta', \gamma+n, \gamma', \mathfrak{x}, y) \cdot s^{\alpha+n} u^{\beta+n} t^{\beta'} p^{\gamma+n} q^{\gamma'}$$

Equating the two values of $(\exp a E_{\alpha\beta\gamma}) f_{\alpha\beta\beta'\gamma\gamma'}$ we get

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', \mathfrak{x} + \text{sup}\, a, y)\, s^{\alpha} u^{\beta} t^{\beta'} p^{\gamma} q^{\gamma'}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\lfloor n} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot F_2(\alpha+n, \beta+n, \beta', \gamma+n, \gamma', \mathfrak{x}, y) \cdot s^{\alpha+n} u^{\beta+n} t^{\beta'} p^{\gamma+n} q^{\gamma'}$$

which finally gives the generating relation

10.12.5) $F_2(\alpha, \beta, \beta', \gamma, \gamma', \mathfrak{x} + \text{sup}\, a, y)$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{(a\,\text{sup})^n}{\lfloor n} \cdot F_2(\alpha+n, \beta+n, \beta', \gamma+n, \gamma', \mathfrak{x}, y)$$

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